Prasse – Theoria Aequationum – – –

USUS

LOGARITHMORUM INFINITINOMII

IN

THEORIA AEQUATIONUM

AUCTORE

Moritz von

MAURICIO DE PRASSE

ADIECTA EST TABULA SINGULARIS

LIPSIAE

APUD CHRIST. THEOPH. RABENHORST.

1796.

§. I.

Institvti Ratio.

Continentur hoc libello Analyseos Combinatoriae *) quaeſtiones tres ex plurimis aliis, quae forma aeque univerſali, ut hic apparent, nusquam, quod equidem ſciam, pertractatae reperiuntur, et quarum inveſtigandarum occaſionem praebuit *mutua relatio exponentium et coëfficientium in aequatione:*

$$[1-(ax+bx^2..)]^{\alpha}[1-(\mathbf{a}x+\mathbf{b}x^2..)]^{\beta}\ldots[1-(\mathfrak{A}x+\mathfrak{B}x^2..)]^{\varepsilon}=1-(Ax+Bx^2+Cx^3\ldots+Nx^n..)$$

Propter *Methodi Combinatoriae* in quaeſtionum analyticarum ſolutione hic adhibitae ipſarumque quaeſtionum indolem viſum eſt mihi, non abs re eſſe, propoſitiones aliquot nondum ſatis notas praeparationis loco praemittere, quibus *formulae dignitatum Polynomii* diverſae et quaedam de *Combinatoriis Jnvolutionibus* hoc in libello obviis, continentur, ut deinde §. XXVII ſeqq. remotis omnibus impedimentis, rei poſſit ſumma afferri ejusque uſus exemplis nonnullis illuſtrari.

Si quid ego effecerim, combinatoriis involutionibus ad Analyſeos theoremata ſummi momenti translatis, eam ob rem ſtudium, quo me disciplinis Matheſeos tradendis dicavi, cuique ut probetur ab illis non alieno, jam in votis habeo: Peritis vero ſatisfacere non, niſi in poſterum, conabor.

§. II.

*) Theoriam artis combinatoriae ejusque ad Analyſin applicationem, Hindenburgius, *Vir Celeberrimus*, in opere, quod inſcribitur: *Novi Syſtematis Permutationum, Combinationum ac Variationum primae lineae* (Lipſiae 1781.), primus ita propoſuit, ut in ea, veluti in fundamento, calculus nitatur omnis. Multa etiam continent ad rem combinatoriam et Analyſin pertinentia, propoſitis ſimul plurimis et exemplis et tabulis combinatoriis, ipſius: *Infinitinomii Dignitatum Hiſtoria, Leges ac Formulae.* . . (Gottingae 1779.). Quibus vero meditationibus peculiaribus, qua via quibusque artificiis ad memorabile illud inventum perductus fuerit, docuit nuper, omnia ſimul enumerans ſcripta eo pertinentia: *Archiv der reinen und angewandten Mathematik*, (Leipzig bey Schäfer 1794.) Heft II. pag. 242.

H.C.M.

A

§. II.

Definitiones.

1) Ars combinatoria res datas ex ordine notat ſive literis *a, b, c, d, e, f, g, h.* . . . . ; ſive numeris 1, 2, 3, 4, 5, 6, 7, 8 . . . . . ; Signa ſingula *Elementorum,* collecta vero $\begin{pmatrix} a, & b, & c, & d, & e, & f & \ldots \\ 1, & 2, & 3, & 4, & 5, & 6 & \ldots \end{pmatrix}$ *Indicis* nomen habent.

2) Elementorum conjunctiones Singulae, veluti *abc*; *ddd*; *aecbd*; ſive 123; 444; 15324; *Complexiones* vocantur, et quidem *rite ordinatae*, in quibus (ſi a ſiniſtra ad dextram legas) elementum poſterius nullum antepoſitum eſt priori, e. g. *aab*; *abc*; *abcd*; 112; 123; 1234; Itaque *aba*; *cab*; *abfee; aaacb*; 121; 312; 12655; 11132; non ſunt rite ordinatae.

3) Complexiones dividuntur

in *Biniones*, quales ſunt *aa*; *ab*; *be*; 11; 12; 23;
" *Terniones*, " " *aaa*; *abc*; *cbd*; 111; 123; 324;
" *Quaterniones* " " *abbe*; *adef*; 1223; 1456;
" " " " " " " " "
" *mtiones* " " *abcd* . . . . *m*; 1234 . . . . *m*;

prout binis, ternis, quaternis, ... *m* Elementis conſtant. Ad hujus appellationis analogiam ſingula elementa ipſa dicuntur *Uniones* v. c. *a*; *b*; *c*; 1; 2; 3;

4) Cum reſpicitur ad aliquam conjungendi legem, ſecundum quam ipſae complexiones producuntur,

collectio Unionum vocatur *prima Claſſis*
" Binionum " *ſecunda* "
" Ternionum " *tertia* "
" " " " "
" *m*tionum " *mta* "

5) *Claſſis rite ordinata* dicitur, cujus complexiones omnes ad inſtar numerorum creſcentium procedunt.

Sic v. c. Complexiones $\begin{pmatrix} 116; & 125; & 134; & 224; & 233 \\ aaf; & abe; & acd; & bbd; & bcc \end{pmatrix}$; non autem $\begin{pmatrix} 224; & 134; & 233; & 116; & 125 \\ bbd; & acd; & bcc; & aaf; & abe \end{pmatrix}$ aut aliter diſpoſitae, ut numeri creſcentes, progrediuntur.

6) *Diſcerptiones numeri n* eae vocantur complexiones numericae, in quibus *elementorum* cujusque complexionis *ſumma* *) aequat numerum *n*.

7) Com-

*) In complexione *ſumma* elementorum cum ipſorum *numero* confundendus non eſt. Illa ab ipſorum elementorum numericorum *magnitudine*, hic a multitudine pendet, et in eadem claſſe conſtanter

7) *Combinationes summae propositae n* eae dicuntur complexiones numericae rite ordinatae (Def. 2), quae simul discerptiones sunt numeri *n* (Def. 6); quo pertinent v. c. summae 5 combinationes a latere collocatae et per classes (Def. 4.) dispositae.

5
14 23
113 122
1112
11111

8) *Permutari* datae combinationis elementa dicuntur, ubi omnibus, quibus possunt, modis sedibus transponuntur suis, et prodeuntes permutando conjunctiones ipsae *Permutationes datae combinationis* vocantur.

§. III.

Problema.

Data Combinatione Summae n (§. II. Def. 7.) *reperire proxime sequentem Classis rite ordinatae* (§. II. Def. 5) et *indicis* (1, 2, 3, 4 ....) (§. Def. 1).

Combinatio data sit v. c. 112445; ubi $n = 17$ et index (1, 2, 3, 4, 5, 6 ....)

Solutio.

I. In data Combinatione quaeratur, a dextra ad sinistram eundo, numerus primus, qui *duabus saltem unitatibus* differat a numero extremo ad dextram et *inventus* (hic 2) *unitate augeatur*. Si talis numerus non reperitur, combinatio data ipsa est classis suae ultima, veluti 233333.

II. Numerus secundum I. auctus (jam 3) in omnibus ad dextram sedibus (quae adsunt) collocetur, excepta tamen extrema.

III. Numeri ad sinistram (si qui adsunt) maneant immutati.

IV. In sede ad dextram extrema ponatur complementum summae *n*. (h. l. complementum est 6)

Vnde

stanter idem est. Sic in complexione 13158 *numerus* elementorum est 5, propter quinque Elementa 1; 3; 1; 5; 8; Elementorum vero *summa* est $1+3+1+5+8 = 18$. In complexione 25412 numerus elementorum iterum est 5, summa autem 14.

Vnde fit:

	ex	112445		ex	113336		ex	113345
Secundum	I.	..3...	Secundum	I.	....4.	Secundum	I.	...4..
"	I. II.	..333.	"	I. II.	....4.	"	I. II.	...44.
"	I. II. III.	11333.	"	I. II. III.	11334.	"	I. II. III.	113444
"	I. II. III. IV.	113336	"	I. II. III. IV.	113345	"	I. II. III. IV.	113444

Demonstratio.

Patet, illis regulis Discerptionem reperiri. Discerptio vero rite ordinata est, quod numerum auctum sequuntur numeri nulli ipso minores (reg. III. et IV.). Combinationem denique prodire datae proximam, intelligitur eo, quod ultimam sedem tenet numerus maximus eorum, quos, ubi regulae I. satisfactum est, poni licuisset.

§. IV.

Problema.

Combinationum summae propositae n classem quamlibet, v. c. mtam (vbi $m < n$) construere, dato Indice (1, 2, 3, 4 . . . .).

Solutio.

1) Scribantur ($m - 1$) unitates, alia juxta aliam, et vltimo loco complementum ad Summam n; i. e. ($n - m + 1$).

2) Ex prima hac combinatione per regulas §. III. deducatur secunda, ex secunda tertia atque ita quaelibet posterior ex proxime priori, donec istae regulae amplius adhiberi nequeant, atque omnes Combinationes expressae erunt.

Exemplum. Si $n = 10$, $m = 5$, producitur

secundum 1) prima combinatio 1 1 1 1 6

" 2) successive reliquae omnes
$$\begin{cases} 1\ 1\ 1\ 2\ 5 \\ 1\ 1\ 1\ 3\ 4 \\ 1\ 1\ 2\ 2\ 4 \\ 1\ 1\ 2\ 3\ 3 \\ 1\ 2\ 2\ 2\ 3 \\ 2\ 2\ 2\ 2\ 2 \end{cases}$$ *)

§. V.

*) De numerorum discerptionibus egerunt Hindenburgius, Vir Celeberrimus, (*Infinit. Dignitatum Hist. Leg. ac Form.* p. 73. seqq. et p. 129. seqq. et *Progr.* quo *Terminorum ab infinitinomii dignitatibus Coefficientes* Moivraeanos *sequi ordinem lexicographicum ostenditur* Lipsiae 1795.) et Toepferus, Vir Clarissimus, (*Combinatorische Analytik* etc. p. 68.)

§. V.

Explicatio.

Combinationum ſummae propoſitae *n* claſſes (§. II. Def. 7.) ex ordine per literas majores Latinas A, B, C, D, ... et numerum *n*, ſequentem in modum, notantur: Sit v. c. ſumma propoſita $n = 8$, notabitur

prima claſſis combinationum ſummae 8 i. e.	8	Signo	${}^{8}A$
ſecunda	17, 26, 35, 44	.	${}^{8}B$
tertia	116, 125, 134, 224, 233	.	${}^{8}C$
quarta	1115, 1124, 1133, 1223, 2222	.	${}^{8}D$
quinta	11114, 11123, 11222	.	${}^{8}E$
ſexta	111113, 111122	.	${}^{8}F$
ſeptima	1111112	.	${}^{8}G$
octava	11111111	.	${}^{8}H$

Index eſt $(1, 2, 3, 4, 5, 6, 7, 8)$

Numerus *n* (hic 8), qui literae majori hic iungitur a laeva, dicitur *Exponens ſummae* (der Summenexponent). Literae cum numeris *Signa Claſſium* conſtituunt.

Scholion I. Index ſignis combinatoriis ſemper adiiciendus, quod, illo omiſſo, combinatoria ſigna ipſa intelligi nequeunt.

Scholion II. Signa ${}^{n}M_{(1,2,3\ldots n)}$ et ${}^{n}\mathcal{M}_{(1,2,3\ldots n)}$ diligenter diſtinguenda ſunt. Illud enim *claſſem* combinationum ſummae *n duodecimam*, quia M duodecima eſt litera alphabeti A, B, C, D, . . ., hoc vero, in quo $\mathcal{M}$ alterius eſt alphabeti, *mtam* generaliter *claſſem exprimit*.

§. VI.

§. VI.

Definitio.

Combinationes summae propositae n ex elementis literalibus eae vocantur Complexiones, quae ex numericis ejusdem summae, secundum problema §. IV. productis, oriuntur, si loco numerorum (1, 2, 3, 4 ....) literae his respondentes ponantur, hoc modo:

$6 = {}^6A$	$f = {}^6A$	$g = {}^6A$
$\left.\begin{matrix}15\\24\\33\end{matrix}\right\} = {}^6B$	$\left.\begin{matrix}ae\\bd\\cc\end{matrix}\right\} = {}^6B$	$\left.\begin{matrix}bf\\ce\\dd\end{matrix}\right\} = {}^6B$
$\left.\begin{matrix}114\\123\\222\end{matrix}\right\} = {}^6C$	$\left.\begin{matrix}aad\\abc\\bbb\end{matrix}\right\} = {}^6C$	$\left.\begin{matrix}bbe\\bcd\\ccc\end{matrix}\right\} = {}^6C$
$\left.\begin{matrix}1113\\1122\end{matrix}\right\} = {}^6D$	$\left.\begin{matrix}aaac\\aabb\end{matrix}\right\} = {}^6D$	$\left.\begin{matrix}bbbd\\bbcc\end{matrix}\right\} = {}^6D$
$11112 = {}^6E$	$aaaab = {}^6E$	$bbbbc = {}^6E$
$111111 = {}^6F$	$aaaaaa = {}^6F$	$bbbbbb = {}^6F$
$(1, 2, 3, 4, 5, 6)$	$\binom{a, b, c, d, e, f}{1, 2, 3, 4, 5, 6}$	$\binom{b, c, d, e, f, g}{1, 2, 3, 4, 5, 6}$

ubi indices inferius hic appositi ostendunt, ad quos numeros literae pertinent. Hinc intelligitur, quid futurum esset, si indices fuissent

$$\binom{c, d, e, f, g, h}{1, 2, 3, 4, 5, 6} \text{ vel } \binom{d, e, f, g, h, i}{1, 2, 3, 4, 5, 6}$$

§. VII.

Explicatio.

Praefixae signis classium (§. V.) homonymae literae Germanicae minores, veluti

$$\mathfrak{a}^n A;\ \mathfrak{b}^n B;\ \mathfrak{c}^n C;\ \mathfrak{d}^n D;\ \mathfrak{e}^n E;\ \ldots\ldots\ldots\ \mathfrak{m}^n \mathfrak{M}$$

$$\binom{b,\ c,\ d,\ e,\ f,\ g\ \ldots\ldots}{1,\ 2,\ 3,\ 4,\ 5,\ 6\ \ldots\ldots}$$

indicant, singulis Complexionibus literalibus praeponendum esse *Numerum permutationum* *), quo scilicet docetur, quoties complexionis elementa suis transponi queant sedibus (Def. 8.). Sic, posito $n = 8$, est:

$\mathfrak{b}^8 D =$

*) Numerus permutationum complexionis $b\ c\ d\ e\ f\ g\ h \ldots\ldots$; ubi elementa numero m sunt omnia diversa, est $m.\ m-1.\ m-2 \ldots\ldots 3.\ 2.\ 1.$ Quod si autem inter haec aliqua sunt eadem numero β et alia numero γ et alia numero δ etc. fit numerus permutationum

$=$

$$\underset{\left(\begin{smallmatrix} b, & c, & d, & e, & f \ldots \\ 1, & 2, & 3, & 4, & 5 \ldots \end{smallmatrix}\right)}{b^6 D} = \left\{\begin{matrix} 4bbbf \\ 12bbce \\ 6bbdd \\ 12bccd \\ 1cccc \end{matrix}\right\} \quad \text{et} \quad \underset{\left(\begin{smallmatrix} b, & c, & d, & e \ldots \\ 1, & 2, & 3, & 4 \ldots \end{smallmatrix}\right)}{e^8 E} = \left\{\begin{matrix} 5bbbbe \\ 20bbbcd \\ 10bbccc \end{matrix}\right\}$$

§. VIII.

Theorema.

Si $y = bx + cx^2 + dx^3 + ex^4 + \ldots\ldots$, potentiae integrae positivae hujus seriei, signis §. VII. explicatis expressae, ex ordine sunt:

$$\begin{aligned}
y^1 &= \mathfrak{a}^1 \mathrm{A} x + \mathfrak{a}^2 \mathrm{A} x^2 + \mathfrak{a}^3 \mathrm{A} x^3 + \mathfrak{a}^4 \mathrm{A} x^4 + \mathfrak{a}^5 \mathrm{A} x^5 + \mathfrak{a}^6 \mathrm{A} x^6 + \ldots\ldots + \mathfrak{a}^n \mathrm{A} x^n + \ldots \\
y^2 &= \mathfrak{b}^2 \mathrm{B} x^2 + \mathfrak{b}^3 \mathrm{B} x^3 + \mathfrak{b}^4 \mathrm{B} x^4 + \mathfrak{b}^5 \mathrm{B} x^5 + \mathfrak{b}^6 \mathrm{B} x^6 + \ldots\ldots + \mathfrak{b}^n \mathrm{B} x^n + \mathfrak{b}^{n+1} \mathrm{B} x^{n+1} + \ldots \\
y^3 &= \mathfrak{c}^3 \mathrm{C} x^3 + \mathfrak{c}^4 \mathrm{C} x^4 + \mathfrak{c}^5 \mathrm{C} x^5 + \mathfrak{c}^6 \mathrm{C} x^6 + \ldots\ldots + \mathfrak{c}^n \mathrm{C} x^n + \mathfrak{c}^{n+1} \mathrm{C} x^{n+1} + \mathfrak{c}^{n+2} \mathrm{C} x^{n+2} + \ldots \\
&\text{〃} \quad \text{〃} \quad \text{〃} \quad \text{〃} \quad \text{〃} \quad \text{〃} \quad \text{〃} \quad \text{〃} \quad \text{〃} \\
y^m &= \mathfrak{m}^m \mathcal{M} x^m + \mathfrak{m}^{m+1} \mathcal{M} x^{m+1} + \mathfrak{m}^{m+2} \mathcal{M} x^{m+2} + \ldots\ldots + \mathfrak{m}^n \mathcal{M} x^n \ldots\ldots + \mathfrak{m}^{n+m-1} \mathcal{M} x^{n+m-1} \ldots \\
&\qquad\qquad \left(\begin{matrix} b, & c, & d, & e, & f, & g & \ldots\ldots \\ 1, & 2, & 3, & 4, & 5, & 6 & \ldots\ldots \end{matrix}\right)
\end{aligned}$$

Termini hic simul apparent generales dignitatum y^1, y^2, y^3, ... y^m.

Hujus Theorematis demonstratio extat in *Infinit. Dign.* §. XXIII. Cf. *Nov. Syst.* p. LIV. g.

§. IX.

Problema.

Dignitatis y^m §. VIII. terminum construere quemlibet independenter v. c. posito $m = 4$, dignitatis y^4 terminum septimum.

Solu-

$= \frac{m.\, m-1.\, m-2.\, m-3 \ldots\ldots 3.\, 2.\, 1.}{1.\, 2.\, 3. \ldots \beta.\, 1.\, 2.\, 3. \ldots \gamma.\, 1.\, 2.\, 3. \ldots \delta \text{ etc.}}$ Hinc numerus permutationum combinationis $bcdefg$ est $6.\, 5.\, 4.\, 3.\, 2.\, 1 = 720$ complexionis autem $b^2 c^3 d = \frac{6.\, 5.\, 4.\, 3.\, 2.\, 1}{1.\, 2.\, 1.\, 2.\, 3.\, 1} = 60$. (Cf. Hindenburgii *Infinit. Dign.* §. XIII. p. 32. *Nov. Syst. Perm.* p. 24. *KAESTNERI, VIRI SVMMI, Anal. endl. Grössen* ed. 1794. §. 34.) Hunc numerum (combinatorium scilicet origine), in complexionibus formularum polynomii occurrentem, *coefficientem polynomialem* dixit Hindenburgius (*Nov. Syst.* p. IX.) ad analogiam Coefficientis binomialis, quocum etiam saepissime conjungitur.

Solutio.

Ex termino nto $\underset{\binom{b,\,c,\,d\,\ldots}{1,\,2,\,3\,\ldots}}{m^{n+m-1}\mathfrak{M}}\, x^{n+m-1}$ seriei y^m (§. VIII.) sequitur, posito $n=7$ et $m=4$, dignitatis y^4 terminus septimus $\underset{\binom{b,\,c,\,d\,\ldots}{1,\,2,\,3\,\ldots}}{\mathfrak{b}\,{}^{10}D}\, x^{10}$. Itaque, ut producatur terminus quaesitus,

I. Discerptionum numeri 10. classis construatur quarta secundum §. IV.	II. Numerorum loco ponantur literae respondentes (§. VI.) secundum Indicem $\binom{b,c,d,e,f,g\,..}{1,2,3,4,5,6\,..}$	III. Complexionibus literalibus praefigantur numeri permutationum (Cf. §. VII. *) et singulis potestas x^{10} jungatur.
1117	bbbh	4bbbh x^{10}
1126	bbcg	12bbcg
1135	bbdf	12bbdf
1144	bbee	6bbee
1225	bccf	12bccf
1234	bcde	24bcde
1333	bddd	4bddd
2224	ccce	4ccce
2233	ccdd	6ccdd

Eodem modo terminus quilibet potestatis y^4 in schemate sequenti reperitur:

	(4)	(5)	(6)	(7)	(8)	(9)
Fit secundum I.	1111	1112	1113	1114	1115	1116
			1122	1123	1124	1125
				1222	1133	1134
					1223	1224
					2222	1233
						2223
secundum II.	bbbb	bbbc	bbbd	bbbe	bbbf	bbbg
			bbcc	bbcd	bbce	bbcf
				bccc	bbdd	bbde
					bccd	bcce
					cccc	bcdd
						cccd

Secun-

secundum III. $y^4 = 1bbbb\,x^4 + 4bbbc\,x^5 + \left.\begin{matrix}4bbbd\\6bbcc\end{matrix}\right| x^6 + \left.\begin{matrix}4bbbe\\12bbcd\\4bccc\end{matrix}\right| x^7 + \left.\begin{matrix}4bbbf\\12bbce\\6bbdd\\12bccd\\1cccc\end{matrix}\right| x^8 + \left.\begin{matrix}4bbbg\\12bbcf\\12bbde\\12bcce\\12bcdd\\4cccd\end{matrix}\right| x^9 + \ldots$

Scholion. Terminos quaesitos integros dignitatis y^4 seorsim h. l. scripsimus idque fecimus majoris perspicuitatis gratia, sed non opus est hac distinctione, quia statim potest terminus quilibet per complexiones in II. exhiberi, addendo numeros permutationum complexionibus singulis a laeva, variabilis autem x potestatem respondentem a dextra. Exercitatus paululum poterit etiam, praetereundo I. illico scribere complexiones literales in II. et inde conficere *terminum quaesitum* (III.), qua re totum negotium et contrahitur et sublevatur.

§. X.

Explicatio.

Coefficientes binomiales, cujuslibet exponentis v, sequentem in modum notantur:

$^{v}\mathfrak{A}$ exprimit $\frac{v}{1}$

$^{v}\mathfrak{B}$ " $\frac{v.(v-1)}{1.\ 2}$

$^{v}\mathfrak{C}$ " $\frac{v.(v-1).(v-2)}{1.\ 2.\ 3}$

$^{v}\mathfrak{D}$ " $\frac{v.(v-1).(v-2).(v-3)}{1.\ 2.\ 3.\ 4.}$

" " " "

$^{v}\mathfrak{M}$ " $\frac{v.(v-1).(v-2)\ldots(v-m+1)}{1.\ 2.\ 3.\ldots\ldots m}$

Scholion I. Liquet m nullum hic nisi integrum positivum significare posse numerum. Coefficiens binomialis *generalis*, hoc loco $m$tus, litera alterius alphabeti notatur, quo cognoscatur, eum esse generalem.

Scholion II. Jam signa quoque sequentia ex signis §. X et VII. composita facile intelligentur. v. c.

$$\underset{(b,\,c,\,d,\,e,\,\ldots)}{\overset{}{}}\!{}^{6}\mathfrak{C}\,c\,{}^{8}C \atop {\left(\begin{matrix}b,\,c,\,d,\,e,\,\ldots\\1,\,2,\,3,\,4,\,\ldots\end{matrix}\right)}$$

quod jubet, construi tertiam Classem combinationum summae 8 ex elementis $(b, c, d, e\ldots)$, quae est

$$\begin{matrix}{}^{8}C\\\left(\begin{matrix}b,\,c,\,d,\,e,\,\ldots\\1,\,2,\,3,\,4,\,\ldots\end{matrix}\right)\end{matrix} = \left\{\begin{matrix}b^2g\\bcf\\bde\\c^2e\\cd^2\end{matrix}\right.$$

dein numerum permutationum cuivis complexioni praeponi, fit

 c³C

$$\underset{\left(\begin{smallmatrix} b, & c, & d, & e \ldots \\ 1, & 2, & 3, & 4 \ldots \end{smallmatrix}\right)}{c^8C} = \left[\begin{array}{l} 3b^2g \\ 6bcf \\ 6bde \\ 3c^2e \\ 3cd^2 \end{array}\right]$$

complexionum denique summam multiplicari per productum ${}^6\mathfrak{C} = \frac{6.5.4}{1.2.3}$; quo producitur:

$$\underset{\left(\begin{smallmatrix} b, & c, & d, & e \ldots \\ 1, & 2, & 3, & 4 \ldots \end{smallmatrix}\right)}{{}^6\mathfrak{C}\, c^8C} = \frac{6.5.4}{1.2.3} \left[\begin{array}{l} + 3b^2g \\ + 6bcf \\ + 6bde \\ + 3c^2e \\ + 3cd^2 \end{array}\right]$$

§. XI.

Theorema.

Sit $az^{\mu} + bz^{\mu+\delta} + cz^{\mu+2\delta}\, dz^{\mu+3\delta} + \ldots\ldots = \mathfrak{p}$.

$$\textit{Erit}\ \mathfrak{p}^{\nu} = a^{\nu}z^{\nu\mu} + {}^{\nu}\mathfrak{A}a^{\nu-1}\mathfrak{a}^1 A z^{\nu\mu+\delta} + \left.\begin{array}{l} {}^{\nu}\mathfrak{A}a^{\nu-1}\mathfrak{a}^2A \\ + {}^{\nu}\mathfrak{B}a^{\nu-2}\mathfrak{b}^2B \end{array}\right| z^{\nu\mu+2\delta} + \left.\begin{array}{l} {}^{\nu}\mathfrak{A}a^{\nu-1}\mathfrak{a}^3A \\ + {}^{\nu}\mathfrak{B}a^{\nu-2}\mathfrak{b}^3B \\ + {}^{\nu}\mathfrak{C}a^{\nu-3}\mathfrak{c}^3C \end{array}\right| z^{\nu\mu+3\delta} \ldots + \left.\begin{array}{llll} {}^{\nu}\mathfrak{A} & a^{\nu-1} & \mathfrak{a} & {}^{n}A \\ + {}^{\nu}\mathfrak{B} & a^{\nu-2} & \mathfrak{b} & {}^{n}B \\ + {}^{\nu}\mathfrak{C} & a^{\nu-3} & \mathfrak{c} & {}^{n}C \\ & \vdots & & \\ + {}^{\nu}\mathfrak{M} & a^{\nu-m} & \mathfrak{m}^n & \mathcal{M} \\ & \vdots & & \\ + {}^{\nu}\mathfrak{N} & a^{\nu-n} & \mathfrak{n}^n & \mathcal{N} \end{array}\right| z^{\nu\mu+n\delta} \ldots$$

$$\left(\begin{smallmatrix} b, & c, & d, & e, & f, & \ldots\ldots \\ 1, & 2, & 3, & 4, & 5, & \ldots\ldots \end{smallmatrix}\right)$$

ubi μ, δ et ν numeri quicunque positivi, negativi, integri, fracti esse possunt.

Demonstratio.

Adhibitis signis coefficientium binomialium (§. X.), fit binomium

$$(a + y)^{\nu} = a^{\nu} + {}^{\nu}\mathfrak{A}a^{\nu-1}y^1 + {}^{\nu}\mathfrak{B}a^{\nu-2}y^2 + {}^{\nu}\mathfrak{C}a^{\nu-3}y^3 + {}^{\nu}\mathfrak{D}a^{\nu-4}y^4 + \ldots\ldots$$

Jam, si ponitur

$$y = bx + cx^2 + dx^3 + ex^4 + fx^5 + \ldots\ldots$$

facta substitutione potentiarum hujus seriei, §. VIII. exhibitarum, sequitur:

$(a + bx$

$$
\begin{array}{l|l}
(a+bx+cx^2+dx^3+\ldots)^\nu = a^\nu + {}^\nu\mathfrak{A}\, a^{\nu-1}\, y^1 & = a^\nu + {}^\nu\mathfrak{A}\, a^{\nu-1}\,[\mathfrak{a}^1 \mathrm{A} x + \mathfrak{a}^2 \mathrm{A} x^2 + \mathfrak{a}^3 \mathrm{A} x^3 + \ldots + \mathfrak{a}^n \mathrm{A} x^n + \ldots] \\
\qquad + {}^\nu\mathfrak{B}\, a^{\nu-2}\, y^2 & \quad + {}^\nu\mathfrak{B}\, a^{\nu-2}\,[\mathfrak{b}^2 \mathrm{B} x^2 + \mathfrak{b}^3 \mathrm{B} x^3 + \ldots + \mathfrak{b}^n \mathrm{B} x^n + \ldots] \\
\qquad + {}^\nu\mathfrak{C}\, a^{\nu-3}\, y^3 & \quad + {}^\nu\mathfrak{C}\, a^{\nu-3}\,[\mathfrak{c}^3 \mathrm{C} x^3 + \ldots + \mathfrak{c}^n \mathrm{C} x^n + \ldots] \\
\qquad \vdots & \quad \vdots \\
\qquad + {}^\nu\mathfrak{M}\, a^{\nu-m}\, y^m & \quad + {}^\nu\mathfrak{M}\, a^{\nu-m}\,[\mathfrak{m}^m \mathcal{M} x^m + \ldots + \mathfrak{m}^n \mathcal{M} x^n \ldots] \\
\qquad \vdots & \quad \vdots \\
\qquad + {}^\nu\mathfrak{N}\, a^{\nu-n}\, y^n & \quad + {}^\nu\mathfrak{N}\, a^{\nu-n}\,[\mathfrak{n}^n \mathcal{N} x^n + \ldots] \\
\qquad \vdots & \quad \vdots
\end{array}
\qquad \begin{pmatrix} b, c, d, e, f, \ldots \\ 1, 2, 3, 4, 5, \ldots \end{pmatrix}
$$

terminisque secundum potentias quantitatis x digestis:

$$(a+bx+cx^2+dx^3+\ldots)^\nu =$$

$$
a^\nu + {}^\nu\mathfrak{A} a^{\nu-1} \mathfrak{a}^1 \mathrm{A} x^1
+ \left.\begin{array}{l} {}^\nu\mathfrak{A} a^{\nu-1} \mathfrak{a}^2 \mathrm{A} \\ + {}^\nu\mathfrak{B} a^{\nu-2} \mathfrak{b}^2 \mathrm{B} \end{array}\right| x^2
+ \left.\begin{array}{l} {}^\nu\mathfrak{A} a^{\nu-1} \mathfrak{a}^3 \mathrm{A} \\ + {}^\nu\mathfrak{B} a^{\nu-2} \mathfrak{b}^3 \mathrm{B} \\ + {}^\nu\mathfrak{C} a^{\nu-3} \mathfrak{c}^3 \mathrm{C} \end{array}\right| x^3
+ \ldots\ldots
+ \left.\begin{array}{l} {}^\nu\mathfrak{A} a^{\nu-1} \mathfrak{a}^n \mathrm{A} \\ + {}^\nu\mathfrak{B} a^{\nu-2} \mathfrak{b}^n \mathrm{B} \\ + {}^\nu\mathfrak{C} a^{\nu-3} \mathfrak{c}^n \mathrm{C} \\ \vdots \\ + {}^\nu\mathfrak{M} a^{\nu-m} \mathfrak{m}^n \mathcal{M} \\ \vdots \\ + {}^\nu\mathfrak{N} a^{\nu-n} \mathfrak{n}^n \mathcal{N} \end{array}\right| x^n + \ldots
$$

$$\begin{pmatrix} b, & c, & d, & e, & f, \ldots \\ 1, & 2, & 3, & 4, & 5, \ldots \end{pmatrix}$$

vnde formula prodit Theorematis, posito utrinque $x = z^3$, additoque factore $z^{\nu\pi}$. (HINDENBURGIUS *Nov. Syst.* p. LIV. 7.)

Scholion. ***Formulae dignitatum Infinitinomii***, his et §. VIII. propositae, vocantur *combinatoriae*, quod signis utuntur combinatoriis. Pluribus terminis formula theorematis exhibetur in tabula adjecta.

§. XII.

PROBLEMA.

Dignitatis $\mathfrak{p}^\nu$ (§. XI.) *construere terminum quemlibet independenter, v. c. undecimum.*

SOLUTIO.

SOLUTIO.

10	$+{}^{r}\mathfrak{A}a^{r-1}$	$[\,1l\,]$	$={}^{r}\mathfrak{A}a^{r-1}\,\mathfrak{a}^{10}A$	z^{rn+10}
19		$2bk$		
28		$2ci$		
37	$+{}^{r}\mathfrak{B}a^{r-2}$	$2dh$	$={}^{r}\mathfrak{B}a^{r-2}\,\mathfrak{b}^{10}B$	
46		$2eg$		
55		$1ff$		
118		$3bbi$		
127		$6bch$		
136		$6bdg$		
145	$+{}^{r}\mathfrak{C}a^{r-3}$	$6bef$	$={}^{r}\mathfrak{C}a^{r-3}\,\mathfrak{c}^{10}C$	
226		$3ccg$		
235		$6cdf$		
244		$3cee$		
334		$3dde$		
1117		$4bbbh$		
1126		$12bbcg$		
1135		$12bbdf$		
1144		$6bbee$		
1225	$+{}^{r}\mathfrak{D}a^{r-4}$	$12bccf$	$={}^{r}\mathfrak{D}a^{r-4}\,\mathfrak{d}^{10}D$	
1234		$24bcde$		
1333		$4bddd$		
2224		$4ccce$		
2233		$6ccdd$		
11116		$5bbbbg$		
11125		$20bbbcf$		
11134		$20bbbde$		
11224	$+{}^{r}\mathfrak{E}a^{r-5}$	$30bbcce$	$={}^{r}\mathfrak{E}a^{r-5}\,\mathfrak{e}^{10}E$	
11233		$30bbcdd$		
12223		$20bcccd$		
22222		$1ccccc$		
111115		$6bbbbbf$		
111124		$30bbbbce$		
111133	$+{}^{r}\mathfrak{F}a^{r-6}$	$15bbbbdd$	$={}^{r}\mathfrak{F}a^{r-6}\,\mathfrak{f}^{10}F$	
111223		$60bbbccd$		
112222		$15bbcccc$		
1111114		$7bbbbbbe$		
1111123	$+{}^{r}\mathfrak{G}a_{r-7}$	$42bbbbbcd$	$={}^{r}\mathfrak{G}a^{r-7}\,\mathfrak{g}^{10}G$	
1111222		$35bbbbccc$		
11111113	$+{}^{r}\mathfrak{H}a^{r-8}$	$8bbbbbbbd$	$={}^{r}\mathfrak{H}a^{r-8}\,\mathfrak{h}^{10}H$	
11111122		$28bbbbbbcc$		
111111112	$+{}^{r}\mathfrak{I}a^{r-9}$	$[\,9bbbbbbbbc\,]$	$={}^{r}\mathfrak{I}a^{r-9}\,\mathfrak{i}^{10}I$	
1111111111	$+{}^{r}\mathfrak{K}a^{r-10}$	$[\,1bbbbbbbbbb\,]$	$={}^{r}\mathfrak{K}a^{r-10}\,\mathfrak{k}^{10}K$	

1) Numerus termini quaesiti (h. l. 11) unitate minuatur, discerptionumque numeri residui classes ex ordine omnes construantur (§. IV.)

2) Loco numerorum substituantur (§. VI.) literae respondentes, secundum indicem

$$\begin{pmatrix} b, & c, & d, & e, & f, & g, & h, & i, & k, & l, & \ldots \\ 1, & 2, & 3, & 4, & 5, & 6, & 7, & 8, & 9, & 10, & \ldots \end{pmatrix}$$

3) Complexionibus singulis praefigantur numeri permutationum literarum (§. VII.)

4) ${}^{r}\mathfrak{A}a^{r-1}$ praeponatur *primae* classi
${}^{r}\mathfrak{B}a^{r-2}$. *secundae* .
${}^{r}\mathfrak{C}a^{r-3}$. *tertiae* .
${}^{r}\mathfrak{D}a^{r-4}$. *quartae* .
. .
. .
(vide §. X. Scholion *II*)

5) Complexionibus omnibus addatur factor communis z^{rn+10}.

DEMONSTRATIO.

Haec terminorum constructio per partes congruit cum termino generali formulae dignitatum infinitinomii combinatoriae §. XI. Q. E. F.

Scholion. Termini si omnes seriei p^ν evolvuntur eodem modo, quo hic undecimus, prodit *formula dignitatum infinitinomii algebraice expressa.* Hujus termini primi decem in tabula adjecta conspiciuntur, eamque ob caussam hoc loco undecimus evolutus est.

§. XIII.

Explicatio.

Sit $p = az^\mu + bz^{\mu+\delta} + cz^{\mu+2\delta} + \ldots$ uti §. XI, *Coefficientes* dignitatis p^ν ex ordine his etiam notantur signis:

$p^\nu\varkappa 1$ exprimit *coefficientem primum* i.e. a^ν
$p^\nu\varkappa 2$ . . *secundum* . ${}^\nu\mathfrak{A}\, a^{\nu-1}\, \mathfrak{a}^1 A$
$p^\nu\varkappa 3$ . . *tertium* . ${}^\nu\mathfrak{A}\, a^{\nu-1}\, \mathfrak{a}^2 A + {}^\nu\mathfrak{B}\, a^{\nu-2}\, \mathfrak{b}^2 B$
$p^\nu\varkappa 4$ . . *quartum* . ${}^\nu\mathfrak{A}\, a^{\nu-1}\, \mathfrak{a}^3 A + {}^\nu\mathfrak{B}\, a^{\nu-2}\, \mathfrak{b}^3 B + {}^\nu\mathfrak{C}\, a^{\nu-3}\, \mathfrak{c}^3 C$
. . . .
$p\varkappa(m+1)$ . . *(m+1)tum* . ${}^\nu\mathfrak{A}\, a^{\nu-1}\, \mathfrak{a}^m A + {}^\nu\mathfrak{B}\, a^{\nu-2}\, \mathfrak{b}^m B + {}^\nu\mathfrak{C}\, a^{\nu-3}\, \mathfrak{c}^m C \ldots + {}^\nu\mathfrak{M}\, a^{\nu-m}\, \mathfrak{m}^m \mathfrak{M}$
$p[a, b, c, d \ldots]$ $\quad \begin{pmatrix} b, & c, & d, & e, & f, & \ldots \\ 1, & 2, & 3, & 4, & 5, & \ldots \end{pmatrix}$

Litera p scilicet seriem indicat, ν dignitatis ipsius exponentem, $\varkappa$ cum numero adposito coefficientem numero ipsi respondentem. Tale signum e. g. $p^\nu\varkappa(m+1)$ enunciatur hoc modo: *Seriei* p *ad dignitatem exponentis* ν *evectae coefficiens* $(m+1)$ *tus.* Hinc sequitur:

$$p^\nu = p^\nu\varkappa 1\, z^{\nu\mu} + p^\nu\varkappa 2\, z^{\nu\mu+\delta} + p^\nu\varkappa 3\, z^{\nu\mu+2\delta} + p^\nu\varkappa 4\, z^{\nu\mu+3\delta} + \ldots\ldots + p^\nu\varkappa(m+1)\, z^{\nu\mu+m\delta} + \ldots\ldots$$
$$p[\, a, \, b, \, c, \, d, \, e \, \ldots\ldots \,]$$

Haec coefficientium notae dignitatis p^ν *Signa localia* vocantur, formulaeque ex illis compositae, *formulae locales* *). Signis et formulis localibus adpositum signum $p[a, b, c, d, e \ldots]$ dicitur *Scala* eaque simpliciter seriei datae p coefficientes ex ordine indicat. Scala igitur diligenter ab indice (§. II. Def. 1.) distinguenda est; hoc enim docetur, quos ad coefficientes discerptionum classes referantur **).

Scho-

*) Signa localia *coefficientium* et *terminorum integrorum* (ubi ponitur $\urcorner$ pro $\varkappa$, veluti $p^\nu\urcorner(m+1)$) primus usurpavit Hindenburgius (*Infinit. Dign.* p. 71, 3; p. 93 — 98; 136 — 141). Melior, quae hic adhibetur, notatio *Nov. Syst.* p. XXXIII, 2. De signorum *localium* cum *combinatoriis* comparatione *Ib.* p. LI — LIII. *Id. Paral. ad Ser. Revers.* p. VIII.

**) Rothius, *Vir Clarissimus*, in libello cui titulus: *Formulae de serierum reversione demonstratio universalis. Lipsiae* 1793. Scalas introduxit. Signa idem localia in hac dissertatione ingeniose adhibuit, earumque ope formulam de serierum reversione combinatoriam, ab Eschenbachio (1789) propositam, primus et rigorose demonstravit.

Scholion. Formula localis, exempli loco hic proposita, *dignitatum Infinitinomii formula* appellatur *localis*. Haec quoque in tabula adjecta exprimitur.

* * *

Sed haec de Infinitinomio protulisse, sufficiat. Pergimus jam ad ea explicanda, quae rem nostram propius attingunt.

§. XIV.

DEFINITIO.

Variationes summae propositae n nominantur discerptiones numeri *n* (§. II. Def. 6.) sine discrimine omnes, sive sint rite ordinatae, sive non.

§. XV.

EXPLICATIO.

A

1	1	1	1	1
	2	1	1	1
	1	2	1	1
		3	1	1
	1	1	2	1
		2	2	1
		1	3	1
			4	1
	1	1	1	2
		2	1	2
		1	2	2
			3	2
		1	1	3
			2	3
			1	4
				5

In schemate apposito A, Variationum summarum 1, 2, 3, 4, 5 (§. XIV.), Variationes summarum minorum a Variationibus summarum majorum ita involvuntur, ut illa ex his possint exsecari, id quod angulis interjectis docetur, quam ob rem ipsa haec variationum constructio *Involutio* vocatur *Combinatoria* *).

§. XVI.

PROBLEMA.

Ex involutione Variationum n (§. XV.) *construere Involutionem Variationum summae* $(n+1)$.

SOLU-

*) De Involutionum et Evolutionum (quae inter reliquas operationes combinatorias facile principatum obtinent) natura, diversitate et in disquisitionibus analyticis efficacia, summa et utilitate copiose egit HINDENBURGIUS, *Vir Excellentissimus*, (*Archiv der reinen und angewandten Mathematik H.* I. p. 13. seqq. *H.* II. III. et IV. *Programma* supra (§. IV. *) laudatum.)

SOLVTIO.

I. Variationi summae n cuilibet ad dextram unitas adponatur.

II. Earundem Variationum summae n elementa ad dextram extrema unitate augeantur, et, quae ita prodeunt, complexiones complexionibus per I. ortis verticaliter subscribantur.

v. c. posito $n = 1$,

datum est $\begin{array}{r|}1\end{array}$

hinc fit per I $\begin{array}{r|r}1&1\end{array}$

per I. II. $\begin{array}{r|r|}1&1\\&2\end{array}$

posito $n = 2$,

datum est $\begin{array}{r|r|}1&1\\&2\end{array}$

hinc fit per I. $\begin{array}{r|r|r}1&1&1\\&2&1\end{array}$

per I. II. $\begin{array}{r|r|r|}1&1&1\\&2&1\\&1&2\\&&3\end{array}$

posito $n = 3$,

datum est $\begin{array}{r|r|r|}1&1&1\\&2&1\\&1&2\\&&3\end{array}$

hinc fit per I. $\begin{array}{r|r|r|r}1&1&1&1\\&2&1&1\\&1&2&1\\&&3&1\end{array}$

per I. II. $\begin{array}{r|r|r|r|}1&1&1&1\\&2&1&1\\&1&2&1\\&&3&1\\&1&1&2\\&&2&2\\&&1&3\\&&&4\end{array}$

DEMONSTRATIO.

Ponatur solutionem propositam problemati satisfacere, si quaeratur Involutio Variationum summarum 1, 2, 3, . . . p; Variationes autem summae $(p + 1)$ regulis traditis non reperiri omnes, sed unam vel plures deesse. Jam quaelibet deficientium Variationum summae $(p + 1)$ desineret

vel in ipsam unitatem v. c. 321. 121

vel in numerum unitate majorem v. c. 213. . . . . 53.

Si *illud*, deficeret 321. . . 12, si *hoc*, deficeret 213. . . . 52 in Variationibus summae praecedentis p; id quod suppositioni repugnat. Eodem modo ostenditur, eandem complexionem non posse saepius occurere. Constat autem, solutionem satisfacere problemati, posito $p = 1$, et $p = 2$ itaque satisfacit etiam pro Variationibus summae cujuslibet $(p + 1) = 3; 4; 5; \ldots$ quaerendis.

§. XVII.

§. XVII.

Explicatio.

B.

1	1	1	1	1	1	1
	2	1	1	1	1	1
		3	1	1	1	1
		2	2	1	1	1
			4	1	1	1
			3	2	1	1
				5	1	1
			2	2	2	1
				4	2	1
				3	3	1
					6	1
				3	2	2
					5	2
					4	3
						7

In ſchemate B Combinationes ſummarum 1, 2, 3, 4, 5, 6, 7 ita exhibentur, ut elementorum ordo ſit inverſus ipſaeque Combinationes conſtituant *Involutionem Combinatoriam* (Cf. §. XV.)

§. XVIII.

Problema.

Ex Involutione Combinationum ſummae n §. XVII. conſtruere Involutionem Combinationum ſummae $(n+1)$.

Solutio.

I. Cuivis Combinationi ſummae n adponantur a dextra unitas,

II. Vnitate augeantur extrema ad dextram elementa earum complexionum, quae vel duabus terminantur elementis diverſis, vel unione (§. II. Def. 3.) conſtant, et quae ita prodeunt, complexiones complexionibus per I. ortis verticaliter ſubſcribantur,

v. c. poſito $n = 1$,

datum eſt

1

hinc fit per. I

1	1

per I. II.

1	1
	2

poſito $n = 2$,

datum eſt

1	1
	2

hinc fit per I.

1	1	1
	2	1

per I. II.

1	1	1
	2	1
		3

poſito $n = 3$,

datum eſt

1	1	1
	2	1
		3

hinc fit per I.

1	1	1	1
	2	1	1
		3	1

per I. II.

1	1	1	1
	2	1	1
		3	1
		2	2
			4

Solutionis hujus demonstratio eadem est, quae in §. XVI.

Scholion. Involutionum genus ab eo, quod §. XV — XVIII. proposuimus, diversum, hic vero non adhibitum, extat in *Infinit. Dign.* p. 79. 80, et p. 133.

§. XIX.

Explicatio.

Si *Variationes summarum* 1, 2, 3, 4, ... n exhibeantur involutorie (§. XVI.), et numerorum loco ponantur literae $a, b, c, d, \ldots n$, haec Involutio Variationum notetur signo:

$$\underset{\left(\substack{a,\,b,\,c,\,\ldots\, n\\ 1,\,2,\,3,\,\ldots\, n}\right)}{{}^{n}\mathfrak{J}}$$

atque, si cuilibet complexioni numerus primo a laeva suorum elementorum respondens praefigatur, id Variationum genus summae n cum suis *Comitibus numericis* *) exprimatur signo:

$$\underset{\left(\substack{a,\,b,\,c,\,\ldots\, n\\ 1,\,2,\,3,\,\ldots\, n}\right)}{i^{n}\mathfrak{J}}$$

Exempla. Pro summae n valoribus 1, 2, 3, 4, 5 .... prodeunt:

$$\underset{\left(\substack{a\\ 1}\right)}{i^{1}\mathfrak{J}} = \left.\underline{1a}\right|$$ Variationes summae 1 cum suis comitibus,

$$\underset{\left(\substack{a,b\\ 1,2}\right)}{i^{2}\mathfrak{J}} = \begin{array}{r|r|} \underline{1a} & a \\ & 2b \\ \hline \end{array}$$ Variationes summae 2 cum suis comitibus,

$$\underset{\left(\substack{a,b,c\\ 1,2,3}\right)}{i^{3}\mathfrak{J}} = \begin{array}{r|r|r|} \underline{1a} & a & a \\ & \underline{2b} & a \\ & 1a & b \\ & & \underline{3c} \end{array}$$ Variationes summae 3 cum suis comitibus,

$i^{4}\mathfrak{J} =$

*) Scilicet Hindenburgius, *V. Cel.* complexionum literalium *Comites numericos* ab ipsarum numeris permutationum sive *coefficientibus polynomialibus* (Cf. §. VII. *) distinguit, et a quacunque polynomialium modificatione, aut alia lege oriundos numeros *Comites complexionum* nominat.

$$\underset{\binom{a,b,c,d}{1,2,3,4}}{{}^{4}\mathfrak{J}} = \begin{array}{l} 1a\;a\;a\;a \\ 2b\;a\;a \\ 1a\;b\;a \\ 3c\;a \\ 1a\;a\;b \\ 2b\;b \\ 1a\;c \\ 4d \end{array}$$

Variationes summae 4 cum suis comitibus,

$$\underset{\binom{a,b,c,d,e}{1,2,3,4,5}}{{}^{5}\mathfrak{J}} = \begin{array}{l} 1a\;a\;a\;a\;a \\ 2b\;a\;a\;a \\ 1a\;b\;a\;a \\ 3c\;a\;a \\ 1a\;a\;b\;a \\ 2b\;b\;a \\ 1a\;c\;a \\ 4d\;a \\ 1a\;a\;a\;b \\ 2b\;a\;b \\ 1a\;b\;b \\ 3c\;b \\ 1a\;a\;c \\ 2b\;c \\ 1a\;d \\ 5e \end{array}$$

Variationes summae 5 cum suis comitibus,

etc. etc. etc.

§. XX.

Explicatio.

Involútio Combinationum summae n ex elementis $a, b, c, d, \ldots n$ (§. XVIII et §. VI.) notetur (*Arch. der Math. Heft. IV.* p. 417, 418) signo:

$$\underset{\binom{a,b,c,\ldots n}{1,2,3\ldots n}}{{}^{n}\mathrm{J}} = {}^{n}\mathrm{A} + {}^{n}\mathrm{B} + {}^{n}\mathrm{C} + {}^{n}\mathrm{D} + \ldots\ldots + {}^{n}\mathcal{N}$$
$$\binom{a, b, c, d, \ldots\ldots n}{1, 2, 3, 4, \ldots\ldots n}$$

et,

et, si singulis complexionibus praefigatur numerus permutationum, signo:

$$\underset{\binom{a,\,b,\,c,\,..\,n}{1,\,2,\,3\,..\,n}}{\mathfrak{f}^n J} = \underset{\binom{a,\,b,\,c,\,d,\,...\,n}{1,\,2,\,3,\,4,\,...\,n}}{\mathfrak{a}^n A + \mathfrak{b}^n B + \mathfrak{c}^n C + \mathfrak{d}^n D + \ldots\ldots\, \mathfrak{n}^n \mathcal{N}}$$

Si denique numeri Permutationum per summam n multiplicentur omnes, quilibet autem numero elementorum complexionis, cui adpositus est, dividatur, haec Involutio Combinationum summae n ex elementis $a, b, c, d, ,.. n$ cum *numericis Comitibus* suis (Cf. §. XIX. *) exprimatur signo:

$$\underset{\binom{a,\,b,\,c,\,..\,n}{1,\,2,\,3,\,..\,n}}{\mathfrak{f}^n J} = \underset{\binom{a,\,b,\,c,\,d,\,e,\,....\,n}{1,\,2,\,3,\,4,\,......\,n}}{\frac{n}{1}\mathfrak{a}^n A + \frac{n}{2}\mathfrak{b}^n B + \frac{n}{3}\mathfrak{c}^n C + \frac{n}{4}\mathfrak{d}^n D + \ldots + \frac{n}{m}\mathfrak{m}^n \mathcal{M} + \ldots + \frac{n}{n}\mathfrak{n}^n \mathcal{N}}$$

Exempla. Pro summae n valoribus 1, 2, 3, 4, 5, ... prodeunt:

$$\underset{\binom{a}{1}}{\mathfrak{f}^1 J} = 1a = \underset{\binom{a}{1}}{\frac{1}{1}\mathfrak{a}^1 A}$$

$$\underset{\binom{a,\,b}{1,\,2}}{\mathfrak{f}^2 J} = \begin{array}{ll} 1a & |\,a \\ & 2b \end{array} = \underset{\binom{a,\quad b}{1,\quad 2}}{\frac{2}{1}\mathfrak{a}^2 A + \frac{2}{2}\mathfrak{b}^2 B}$$

$$\underset{\binom{a,\,b,\,c}{1,\,2,\,3}}{\mathfrak{f}^3 J} = \begin{array}{lll} 1a & |\,a & |\,a \\ & 3b & |\,a \\ & & 3c \end{array} = \underset{\binom{a,\quad b,\quad c}{1,\quad 2,\quad 3}}{\frac{3}{1}\mathfrak{a}^3 A + \frac{3}{2}\mathfrak{b}^3 B + \frac{3}{3}\mathfrak{c}^3 C}$$

$$\underset{\binom{a,\,b,\,c,\,d}{1,\,2,\,3,\,4}}{\mathfrak{f}^4 J} = \begin{array}{llll} 1a & |\,a & |\,a & |\,a \\ & 4b & |\,a & |\,a \\ & & 4c & |\,a \\ & & 2b & \;\,b \\ & & & 4d \end{array} = \underset{\binom{a,\quad b,\quad c,\quad d}{1,\quad 2,\quad 3,\quad 4}}{\frac{4}{1}\mathfrak{a}^4 A + \frac{4}{2}\mathfrak{b}^4 B + \frac{4}{3}\mathfrak{c}^4 C + \frac{4}{4}\mathfrak{d}^4 D}$$

$$\underset{\binom{a,\,b,\,c,\,d,\,e}{1,\,2,\,3,\,4,\,5}}{\mathfrak{f}^5 J} = \begin{array}{lllll} 1a & |\,a & |\,a & |\,a & |\,a \\ & 5b & |\,a & |\,a & |\,a \\ & & 5c & |\,a & |\,a \\ & & 5b & \;\,b & |\,a \\ & & & 5d & |\,a \\ & & & 5c & \;\,b \\ & & & & 5e \end{array} = \underset{\binom{a,\quad b,\quad c,\quad d,\quad e}{1,\quad 2,\quad 3,\quad 4,\quad 5}}{\frac{5}{1}\mathfrak{a}^5 A + \frac{5}{2}\mathfrak{b}^5 B + \frac{5}{3}\mathfrak{c}^5 C + \frac{5}{4}\mathfrak{d}^5 D + \frac{5}{5}\mathfrak{e}^5 E}$$

Scholion

Scholion. Signa hic et §. XIX. exhibita ſunt *involutoria*, in quibus recta litera J; *Combinationum*, obliqua vero $\mathcal{J}$; *Variationum* involutoriam conſtructionem (§ XV. et XVII.) ſecundum indicem $\begin{pmatrix} a. & b, & c, & d, & e \dots n \\ 1, & 2, & 3, & 4, & 5 \dots n \end{pmatrix}$, litera n in fronte a laeva ſummam et germanicae minores $\mathfrak{i}$ et $\mathfrak{i}$ comites complexionum reſpondentes ſignificant.

§. XXI.

THEOREMA.

$$Log.nat.[1-(ax+bx^2+cx^3+\dots)] = -\left(\frac{{}^{1}\mathfrak{i}Jx^1}{1}+\frac{{}^{2}\mathfrak{i}Jx^2}{2}+\frac{{}^{3}\mathfrak{i}Jx^3}{3}+\frac{{}^{4}\mathfrak{i}Jx^4}{4}\dots+\frac{{}^{n}\mathfrak{i}Jx^n}{n}+\dots\right)$$
$$\begin{pmatrix} a, & b, & c, & d, & \dots\dots n \\ 1, & 2, & 3, & 4, & \dots\dots n \end{pmatrix}$$

DEMONSTRATIO.

$$\text{Sit } Log.nat.[1-(ax+bx^2+cx^3\dots)] = Log.nat.(1-y) = -\left(\frac{y}{1}+\frac{y^2}{2}+\frac{y^3}{3}\dots+\frac{y^n}{n}\dots\right)$$

unde, dignitatibus ſeriei y ſecundum §. VIII. expreſſis atque ita diſpoſitis, ut coefficientes eiusdem potentiae quantitatis x conſtituant ſeriem verticalem, prodit:

$$Log.nat.[1-(ax+bx^2+cx^3+\dots)] = -\left[\frac{\mathfrak{a}^1A}{1}x + \left.\begin{array}{r} \frac{\mathfrak{a}^2A}{1} \\ +\frac{\mathfrak{b}^2B}{2} \end{array}\right| x^2 + \left.\begin{array}{r} \frac{\mathfrak{a}^3A}{1} \\ +\frac{\mathfrak{b}^3B}{2} \\ +\frac{\mathfrak{c}^3C}{3} \end{array}\right| x^3 + \dots\dots + \left.\begin{array}{r} \frac{\mathfrak{a}^nA}{1} \\ +\frac{\mathfrak{b}^nB}{2} \\ +\frac{\mathfrak{c}^nC}{3} \\ \vdots \\ +\frac{\mathfrak{m}^n\mathfrak{M}}{m} \\ \vdots \\ +\frac{\mathfrak{n}^n\mathfrak{N}}{n} \end{array}\right| x^n + \dots\right]$$
$$\begin{pmatrix} a, & b, & c, & d, & e \dots n, & \dots \\ 1, & 2, & 3, & 4, & 5 \dots n, & \dots \end{pmatrix}$$

Coefficiens termini generalis $\left(\frac{\mathfrak{a}^nA}{1}+\frac{\mathfrak{b}^nB}{2}+\frac{\mathfrak{c}^nC}{3}+\dots\dots+\frac{\mathfrak{m}^n\mathfrak{M}}{m}+\dots\dots+\frac{\mathfrak{n}^n\mathfrak{N}}{n}\right)x^n$ ſecundum §. XX. aequalis eſt $\frac{{}^{n}\mathfrak{i}J}{n}$; itaque, valoribus $1, 2, 3, 4, 5, \dots$ ſucceſſive poſitis loco n, aequatio oritur Theorematis.

§. XXII.

§. XXII.

Theorema.

Est $\underset{(a)}{i^1\mathfrak{J}} = 1a$

$$\underset{(a,b)}{i^2\mathfrak{J}} = \underset{(a)}{i^1\mathfrak{J}a} + 2b$$

$$\underset{(a,b,c)}{i^3\mathfrak{J}} = \underset{(a,\ b)}{i^2\mathfrak{J}a + i^1\mathfrak{J}b} + 3c$$

$$\underset{(a,b,c,d)}{i^4\mathfrak{J}} = \underset{(a,\ b,\ c)}{i^3\mathfrak{J}a + i^2\mathfrak{J}b + i^1\mathfrak{J}c} + 4d$$

$$\cdot \qquad \cdot$$

$$\underset{(a,b,c\ldots n)}{i^n\mathfrak{J}} = \underset{\left(a,\ b,\ c,\ d,\ e\ \ldots\ \ldots\ \overset{-m}{n},\ \ldots\ \overset{-3}{n},\ \overset{-2}{n},\ \overset{-1}{n}\right)}{i^{n-1}\mathfrak{J}a + i^{n-2}\mathfrak{J}b + i^{n-3}\mathfrak{J}c + \ldots + i^{n-m}\mathfrak{J}m + \ldots + i^3\mathfrak{J}\overset{-3}{n} + i^2\mathfrak{J}\overset{-2}{n} + i^1\mathfrak{J}\overset{-1}{n}} + nn$$

In his formulis elementa singula $a, b, c, d, \ldots$, signis involutoriis §. XIX. juncta, ad singulas complexiones, quas signa exprimunt, referenda esse, per se clarum est. Literae autem cum numeris supra positis designant,

$\overset{-1}{n}$ elementum $(n-1)$ tum,

$\overset{-2}{n}$ $\quad$ » $\quad$ $(n-2)$ tum,

$\overset{-3}{n}$ $\quad$ » $\quad$ $(n-3)$ tum,

» $\quad$ . . . . . » . . . .

$\overset{-m}{n}$ $\quad$ » $\quad$ $(n-m)$ tum,

ubi primum elementum est a, et ntum n; nempe numeri $-1, -2, -3, \ldots -m$, scripti supra n, *distantiam* exponunt *retrorsum* ab $n$to, et proinde *Distantiae Exponentes* vocantur. Hindenb. *Nov. Syst.* p. XXXVII. seq. Toepf. *Comb. Anal.* p. 164. seq.

Demonstratio.

Theorematis ratio statim ex ipsa constructione Variationum §. XV. et explicatione §. XIX. liquet.

§. XXIII.

Theorema.

Elementis datae combinationis permutatis permutationumque alia alii subscripta, uti mos est, series elementorum verticalis quaelibet, toties unumquodque continet, quoties prima.

De-

Demonstratio.

Elementum quodvis toties eandem tenet ſedem, quoties reliqua permutari poſſunt, id ex ipſa liquet permutationum definitione (§. II. Def. 8), toties igitur in hac redit ſerie verticali, quoties in illa.

Corollarium I.

Significet $\mathfrak{m}$ numerum permutationum combinationis numericae ſummae n claſſis $m$tae, erit $n.\,\mathfrak{m}$ ſumma omnium numerorum, quibus permutationes iſtae conſtant, et cum numerus quilibet toties in hac recurrit ſerie verticali, quoties in illa, ſingulae ſeriei verticalis ſumma eſt $\frac{n}{m}.\,\mathfrak{m}$. Sic:

$$\text{v. c.}\quad \begin{bmatrix} 22233 \\ 22323 \\ 22332 \\ 23223 \\ 23232 \\ 23322 \\ 32223 \\ 32232 \\ 32322 \\ 33222 \end{bmatrix}$$

ſunt permutationes combinationis numericae 22233 (§. II. Def. 8.) in qua $n = 12$, $m = 5$, et $\mathfrak{m} = 10$, unius itaque ſeriei verticalis ſumma eſt: $\frac{12}{5}.\,10 = 2+2+2+2+2+2+3+3+3+3$

Corollarium II.

Elementis datae combinationis Summae n, claſſis $m$tae et indicis $\begin{pmatrix} a, b, c, d, e, \ldots n \\ 1, 2, 3, 4, 5, \ldots \mathrm{n} \end{pmatrix}$ permutatis, et praepoſito cuique permutationi numero, qui primo ad laevam ipſius elemento reſpondet, ea re productum data combinatione expreſſum multiplicatur per $\frac{n}{m}.\,\mathfrak{m}$ ſecundum Corollarium I. Sic data combinatione $bbbcc$ eſt $n = 12$, $m = 5$, $\mathfrak{m} = 10$.

$$\text{atque}\quad \begin{Bmatrix} 2bbbcc \\ 2bbcbc \\ 2bbccb \\ 2bcbbc \\ 2bcbcb \\ 2bccbb \\ 3cbbbc \\ 3cbbcb \\ 3cbcbb \\ 3ccbbb \end{Bmatrix} = \frac{12}{5}.\,10.\,bbbcc = 24\,bbbcc.$$

§. XXIV.

Theorema.

$$\underset{\begin{pmatrix} a, & b, & c, & d, & \ldots\ldots & n \\ 1, & 2, & 3, & 4, & \ldots\ldots & \mathrm{n} \end{pmatrix}}{\mathrm{n}\left[\tfrac{1}{1}\mathfrak{a}^{\mathrm{n}}A+\tfrac{1}{2}\mathfrak{b}^{\mathrm{n}}B+\tfrac{1}{3}\mathfrak{c}^{\mathrm{n}}C+\tfrac{1}{4}\mathfrak{d}^{\mathrm{n}}D+\ldots+\tfrac{1}{m}\mathfrak{m}^{\mathrm{n}}\mathcal{M}+\ldots\tfrac{1}{n}\mathfrak{n}^{\mathrm{n}}\mathcal{N}\right]} = \underset{\begin{pmatrix} a,b,c,\ldots n \\ 1,2,3,\ldots \mathrm{n} \end{pmatrix}}{\mathfrak{i}^{\mathrm{n}}J} = \underset{\begin{pmatrix} a,b,c,\ldots n \\ 1,2,3,\ldots \mathrm{n} \end{pmatrix}}{\mathfrak{f}^{\mathrm{n}}\mathcal{J}}$$

De-

DEMONSTRATIO.

I. Ipsa liquet explicatione §. XX. esse

$$n\left(\frac{1}{1}a^nA+\frac{1}{2}b^nB+\frac{1}{3}c^nC+\frac{1}{4}d^nD+\ldots+\frac{1}{m}m^n\mathcal{M}+\ldots\frac{1}{n}n^n\mathcal{N}\right)_{\binom{a,\ b,\ c,\ d,\ \ldots\ n}{1,\ 2,\ 3,\ 4,\ \ldots\ n}} = i^nJ_{\binom{a,\ b,\ c,\ \ldots\ n}{1,\ 2,\ 3,\ \ldots\ n}}$$

II. Si pro Combinationibus summae n, Classis $m$tae cum numeris permutationum suarum i.e. pro:

$$m^n\mathcal{M}_{\binom{a,\ b,\ c,\ \ldots\ n}{1,\ 2,\ 3,\ \ldots\ n}}$$

ipsae harum Combinationum substituantur permutationes (i.e. Variationum summae n Classis $m$ta Def. §. XIV.) et cuique permutationi praefigatur numerus primo elementorum suorum respondens; eo producta signo:

$${}^n\mathcal{M}_{\binom{a,\ b,\ c,\ \ldots\ n}{1,\ 2,\ 3,\ \ldots\ n}}$$

expressa secundum Corollarium II. §. XXIII. multiplicantur per $\frac{n}{m}.m$ (valor scilicet numeri permutationum m pendet a diversitate factorum cujusvis illorum productorum v. §. VII.*)

Itaque $\frac{n}{1}a^nA+\frac{n}{2}b^nB+\frac{n}{3}c^nC+\frac{n}{4}d^nD+\ldots+\frac{n}{m}m^n\mathcal{M}+\ldots+\frac{n}{n}n^n\mathcal{N}_{\binom{a,\ b,\ c,\ d,\ e,\ \ldots\ n}{1,\ 2,\ 3,\ 4,\ 5,\ \ldots\ n}}$.

Variationibus exprimitur, si prima, secunda, tertia, verbo Classes omnes Variationum summae n construuntur et cuique variationi proponitur numerus primo suorum elementorum respondens. His variationibus autem involutorie dispositis oritur (§. XIX.):

$$i^n\mathcal{J}_{\binom{a,\ b,\ c,\ \ldots\ n}{1,\ 2,\ 3,\ \ldots\ n}} = \frac{n}{1}a^nA+\frac{n}{2}b^nB+\frac{n}{3}c^nC+\frac{n}{4}d^nD+\ldots+\frac{n}{m}m^n\mathcal{M}+\ldots+\frac{n}{n}n^n\mathcal{N}_{\binom{a,\ b,\ c,\ d,\ \ldots\ n}{1,\ 2,\ 3,\ 4,\ \ldots\ n}}$$

Superest ut exempla afferantur:

Posito $n=1$, est $\underline{1a|} = \underline{1a|} = 1\left(\frac{1}{1}a^1A\right)_{\binom{a}{1}}$

$n=2$, $\begin{array}{l|l} 1a & a \\ 2b & \end{array} = \begin{array}{l|l} 1a & a \\ 2b & \end{array} = 2\left(\frac{1}{1}a^2A+\frac{1}{2}b^2B\right)_{\binom{a,\ b}{1,\ 2}}$

$n=3$, $\begin{array}{l|l|l} 1a & a & a \\ 2b & a & \\ 1a & b & \\ 3c & & \end{array} = \begin{array}{l|l|l} 1a & a & a \\ 3b & a & \\ 3c & & \end{array} = 3\left(\frac{1}{1}a^3A+\frac{1}{2}b^3B+\frac{1}{3}c^3C\right)_{\binom{a,\ b,\ c}{1,\ 2,\ 3}}$

Po-

Posito $n = 4$ est $\begin{array}{rlll} 1a| & a| & a| & a| \\ & 2b| & a| & a| \\ & 1a & b| & a| \\ & & 3c| & a| \\ & 1a & a & b| \\ & & 2b & b| \\ & & 1a & c| \\ & & & 4d| \end{array} = \begin{array}{rlll} 1a| & a| & a| & a| \\ & 4b| & a| & a| \\ & & 4c| & a| \\ & & 2b & b| \\ & & & 4d| \end{array} = 4\left(\frac{1}{1}a^4A + \frac{1}{2}b^4B + \frac{1}{3}c^4C + \frac{1}{4}d^4D\right)$

$$\left(\underset{1}{a},\ \underset{2}{b},\ \underset{3}{c},\ \underset{4}{d}\right)$$

§. XXV

Explicatio.

Ponatur:

$${}^1\Sigma = \underset{(a)}{{}^1J}\,\alpha + \underset{(a)}{{}^1J}\,\beta + \dots + \underset{(\mathfrak{A})}{{}^1J}\,\varrho = 1a|\alpha + 1a|\beta + \dots + 1\mathfrak{A}|\varrho$$

$${}^2\Sigma = \underset{(a,b)}{{}^2J}\,\alpha + \underset{(a,b)}{{}^2J}\,\beta + \dots + \underset{(\mathfrak{A},\mathfrak{B})}{{}^2J}\,\varrho = \begin{array}{ll} 1a| & a| \\ & 2b| \end{array}\alpha + \begin{array}{ll} 1a| & a| \\ & 2b| \end{array}\beta + \dots + \begin{array}{ll} 1\mathfrak{A}| & \mathfrak{A}| \\ & 2\mathfrak{B}| \end{array}\varrho$$

$${}^3\Sigma = \underset{(a,b,c)}{{}^3J}\,\alpha + \underset{(a,b,c)}{{}^3J}\,\beta + \dots + \underset{(\mathfrak{A},\mathfrak{B},\mathfrak{C})}{{}^3J}\,\varrho = \begin{array}{lll} 1a| & a| & a| \\ & 3b| & a| \\ & & 3c| \end{array}\alpha + \begin{array}{lll} 1a| & a| & a| \\ & 3b| & a| \\ & & 3c| \end{array}\beta + \dots + \begin{array}{lll} 1\mathfrak{A}| & \mathfrak{A}| & \mathfrak{A}| \\ & 3\mathfrak{B}| & \mathfrak{A}| \\ & & 3\mathfrak{C}| \end{array}\varrho$$

$${}^n\Sigma = \underset{(a,b,\dots n)}{{}^nJ}\,\alpha + \underset{(a,b,\dots n)}{{}^nJ}\,\beta + \dots + \underset{(\mathfrak{A},\mathfrak{B},\dots\mathfrak{N})}{{}^nJ}\,\varrho$$

(Conf. Explicationem §. XX.)

§. XXVI.

Theorema.

$$Log.\,nat.[1-(ax+bx^2..)]^{\alpha}[1-(ax+bx^2..)]^{\beta}..[1-(\mathfrak{A}x+\mathfrak{B}x^2..)]^{\varrho} = \left(\frac{{}^1\Sigma}{1}x + \frac{{}^2\Sigma}{2}x^2 + .. + \frac{{}^n\Sigma}{n}x^n ..\right)$$

Demon-

Demonstratio.

Est enim (§. XXI.), si cuique termino index adponatur:

$$\alpha\, Log.\, nat.[1-(ax+bx^2+cx^3+..)] = -\left(\underset{(a)}{[^1]}\alpha\frac{x}{1}+\underset{(a,b)}{[^2]}\alpha\frac{x^2}{2}+\underset{(a,b,c)}{[^3]}\alpha\frac{x^3}{3}+\dots+\underset{(a,b,..n)}{[^n]}\alpha\frac{x^n}{n}+\dots\right)$$

$$\beta\, Log.\, nat.[1-(\mathfrak{a}x+\mathfrak{b}x^2+\mathfrak{c}x^3+..)] = -\left(\underset{(\mathfrak{a})}{[^1]}\beta\frac{x}{1}+\underset{(\mathfrak{a},\mathfrak{b})}{[^2]}\beta\frac{x^2}{2}+\underset{(\mathfrak{a},\mathfrak{b},\mathfrak{c})}{[^3]}\beta\frac{x^3}{3}+\dots+\underset{(\mathfrak{a},\mathfrak{b},..\mathfrak{n})}{[^n]}\beta\frac{x^n}{n}+\dots\right)$$

. . . .

$$\rho\, Log.\, nat.[1-(\mathfrak{A}x+\mathfrak{B}x^2+\mathfrak{C}x^3+..)] = -\left(\underset{(\mathfrak{A})}{[^1]}\rho\frac{x}{1}+\underset{(\mathfrak{A},\mathfrak{B})}{[^2]}\rho\frac{x^2}{2}+\underset{(\mathfrak{A},\mathfrak{B},\mathfrak{C})}{[^3]}\rho\frac{x^3}{3}+\dots+\underset{(\mathfrak{A},\mathfrak{B},..\mathfrak{N})}{[^n]}\rho\frac{x^n}{n}+\dots\right)$$

unde, si coefficientes earundem potentiarum quantitatis x per omnes series verticales colligantur et signis §. XXV. exprimantur, formula prodit proposita.

§. XXVII.

Haec ubi praemissa sunt, ad ipsius libelli summam convertimur. In aequatione scilicet supra (§. I.) proposita:

$$[1-(ax+bx^2\dots)]^\alpha[1-(\mathfrak{a}x+\mathfrak{b}x^2..)]^\beta\dots[1-(\mathfrak{A}x+\mathfrak{B}x^2..)]^\rho = 1-(\text{A}x+\text{B}x^2+\text{C}x^3..+\text{N}x^n\dots)$$

ubi numerus factorum indeterminatus est atque Coefficientium et Exponentium valores quicunque esse possunt,

I. Summa Involutionum $\underset{(a,b,c..n)}{[^n]}\alpha + \underset{(\mathfrak{a},\mathfrak{b},\mathfrak{c},..\mathfrak{n})}{[^n]}\beta \dots + \underset{(\mathfrak{A},\mathfrak{B},\mathfrak{C}..\mathfrak{N})}{[^n]}\rho$; per Coefficientes A, B, C, . . N;

II. Coefficientes A, B, C, D, E, F, ... N, per Exponentes α, β, γ, δ, ε, ζ, ... ρ; et Coefficientes (a, b, c, d, . .); ($\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$, . . .); . . . ($\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, . . .);

III. Exponentes α, β, γ, δ, . . ρ, per Coefficientes (a, b, c, d, . . .); ($\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$, . .); . . ($\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, . . .); et A, B, C, D, . . . exprimentur.

Harum relationum prima et secunda, quibus multa communia sunt, proximis duobus paragraphis afferuntur, de tertia vero ab illis longe diversa, problemate de Incognitarum Eliminatione interjecto, actum est §. XXXI.

§. XXVIII.

Theorema.

Data aequatione §. XXVII. exposita:

$$[1-(ax+bx^2..)]^{\alpha}[1-(\mathfrak{a}x+\mathfrak{b}x^2..)]^{\beta}..[1-(\mathfrak{A}x+\mathfrak{B}x^2..)]^{\varrho}=1-(Ax+Bx^2+Cx^3\ldots+Nx^n\ldots)$$

sequitur:

$$I.\ {}^{n}\Sigma = \underset{(A,B,C,\ldots N)}{i^{n}J} = \underset{(a,b,\ldots n)}{i^{n}J\,\alpha} + \underset{(\mathfrak{a},\mathfrak{b},\ldots\mathfrak{n})}{i^{n}J\,\beta} + \ldots\ldots + \underset{(\mathfrak{A},\mathfrak{B},\ldots\mathfrak{N})}{i^{n}J\,\varrho}.$$

$$II.\ {}^{n}\Sigma = {}^{n-1}\Sigma A + {}^{n-2}\Sigma B + {}^{n-3}\Sigma C + \ldots + {}^{n-m}\Sigma M + \ldots + {}^{1}\Sigma \overset{-1}{N} + n N.$$

$$= \begin{cases} +\alpha\,[\underset{(a,\ b,\ c,\ \ldots\ldots\ \overset{-1}{n})}{i^{n-1}\mathcal{J}a + i^{n-2}\mathcal{J}b + i^{n-3}\mathcal{J}c + \ldots + i^{n-m}\mathcal{J}m + \ldots + i^{1}\mathcal{J}\overset{-1}{n}} + n n] \\ +\beta\,[\underset{(\mathfrak{a},\ \mathfrak{b},\ \mathfrak{c},\ \ldots\ldots\ \overset{-1}{\mathfrak{n}})}{i^{n-1}\mathcal{J}\mathfrak{a} + i^{n-2}\mathcal{J}\mathfrak{b} + i^{n-3}\mathcal{J}\mathfrak{c} + \ldots + i^{n-m}\mathcal{J}\mathfrak{m} + \ldots + i^{1}\mathcal{J}\overset{-1}{\mathfrak{n}}} + n\mathfrak{n}] \\ \vdots \\ +\varrho\,[\underset{(\mathfrak{A},\ \mathfrak{B},\ \mathfrak{C},\ \ldots\ldots\ \overset{-1}{\mathfrak{N}})}{i^{n-1}\mathcal{J}\mathfrak{A} + i^{n-2}\mathcal{J}\mathfrak{B} + i^{n-3}\mathcal{J}\mathfrak{C} + \ldots + i^{n-m}\mathcal{J}\mathfrak{M} + \ldots + i^{1}\mathcal{J}\overset{-1}{\mathfrak{N}}} + n\mathfrak{N}] \end{cases}$$

Demonstratio.

Secundum §. XXI. est:

$$Log.nat.[1-(Ax+Bx^2+Cx^3+Dx^4..+Nx^n..)] = -\Big(\underset{\left(\begin{smallmatrix}A, & B, & C, & D, & \ldots N\\ 1, & 2, & 3, & 4, & \ldots n\end{smallmatrix}\right)}{\frac{i^{1}Jx^{1}}{1}+\frac{i^{2}Jx^{2}}{2}+\frac{i^{3}Jx^{3}}{3}\ldots\ldots+\frac{i^{n}Jx^{n}}{n}..}\Big)$$

et secundum §. XXVI.

$$Log.nat.[1-(ax+bx^2..)]^{\alpha}[1-(\mathfrak{a}x+\mathfrak{b}x^2..)]^{\beta}..[1-(\mathfrak{A}x+\mathfrak{B}x^2..)]^{\varrho} = -[\frac{{}^{1}\Sigma x}{1}+\frac{{}^{2}\Sigma x^2}{2}\ldots+\frac{{}^{n}\Sigma x^n}{n}..]$$

Itaque, utriusque seriei coefficientibus comparatis, sequitur:

$$I.\ {}^{n}\Sigma = \underset{(A,B,C,\ldots N)}{i^{n}J} = \underset{(a,b,\ldots n)}{i^{n}J\,\alpha} + \underset{(\mathfrak{a},\mathfrak{b},\ldots\mathfrak{n})}{i^{n}J\beta} + \ldots\ldots + \underset{(\mathfrak{A},\mathfrak{B},\ldots\mathfrak{N})}{i^{n}J\varrho}$$

Jam si adhibentur Combinationum loco Variationes, uti traditum est §. XXIV. fit:

$$^{n}\Sigma = \underset{(A,B,C,\ldots N)}{i^{n}\mathcal{J}} = \underset{(a,b,\ldots n)}{i^{n}\mathcal{J}\alpha} + \underset{(\mathfrak{a},\mathfrak{b},\ldots\mathfrak{n})}{i^{n}\mathcal{J}\beta} + \ldots\ldots + \underset{(\mathfrak{A},\mathfrak{B},\ldots\mathfrak{N})}{i^{n}\mathcal{J}\varrho}$$

five

sive, signis involutoriis secundum §. XXII. evolutis:

$${}^{n}\Sigma = \underset{(A,\ B,\ C,\ \ldots\ \overset{-1}{N})}{i^{n-1}\mathcal{J}A + i^{n-2}\mathcal{J}B + i^{n-3}\mathcal{J}C + \ldots + i^{n-m}\mathcal{J}M + \ldots + i^{1}\mathcal{J}\overset{-1}{N}} + nN$$

cumque ex formula *I* et §. XXIV. posito (n-m) loco (n), sequatur:

$${}^{n-m}\Sigma = \underset{(A,B,C..)}{i^{n-m}J} = \underset{(A,B,C..)}{i^{n-m}\mathcal{J}}.$$

Formula prodit universalis, terminis recurrentibus composita:

$$II.\ {}^{n}\Sigma = {}^{n-1}\Sigma A + {}^{n-2}\Sigma B + {}^{n-3}\Sigma C + \ldots + {}^{n-m}\Sigma M + \ldots + {}^{1}\Sigma\overset{-1}{N} + nN$$

$$= \left\{ \begin{array}{l} + \alpha\,[\,\underset{(a,\ b,\ c,\ \ldots\ \overset{-1}{n})}{i^{n-1}\mathcal{J}a + i^{n-2}\mathcal{J}b + i^{n-3}\mathcal{J}c + \ldots + i^{n-m}\mathcal{J}m + \ldots + i^{1}\mathcal{J}\overset{-1}{n}} + nn\,] \\ + \beta\,[\,\underset{(a,\ b,\ c,\ \ldots\ \overset{-1}{n})}{i^{n-1}\mathcal{J}a + i^{n-2}\mathcal{J}b + i^{n-3}\mathcal{J}c + \ldots + i^{n-m}\mathcal{J}m + \ldots + i^{1}\mathcal{J}\overset{-1}{n}} + nn\,] \\ \quad \vdots \\ + \varrho\,[\,\underset{(\mathfrak{A},\ \mathfrak{B},\ \mathfrak{C},\ \ldots\ \overset{-1}{\mathfrak{N}})}{i^{n-1}\mathcal{J}\mathfrak{A} + i^{n-2}\mathcal{J}\mathfrak{B} + i^{n-3}\mathcal{J}\mathfrak{C} + \ldots + i^{n-m}\mathcal{J}\mathfrak{M} + \ldots + i^{1}\mathcal{J}\overset{-1}{\mathfrak{N}}} + n\mathfrak{N}\,] \end{array} \right.$$

Ex formula *I.* sequitur, posito $n = 1;\ 2;\ 3;\ 4;$

$${}^{1}\Sigma = \begin{array}{c|}1A\end{array} = \begin{array}{c|}1a\end{array}\alpha + \begin{array}{c|}1a\end{array}\beta + \ldots\ldots + \begin{array}{c|}1\mathfrak{A}\end{array}\varrho$$

$${}^{2}\Sigma = \begin{array}{c|c|}1A & A \\ & 2B\end{array} = \begin{array}{c|c|}1a & a \\ & 2b\end{array}\alpha + \begin{array}{c|c|}1a & a \\ & 2b\end{array}\beta + \ldots\ldots + \begin{array}{c|c|}1\mathfrak{A} & \mathfrak{A} \\ & 2\mathfrak{B}\end{array}\varrho$$

$${}^{3}\Sigma = \begin{array}{c|c|c|}1A & A & A \\ & 3B & A \\ & & 3C\end{array} = \begin{array}{c|c|c|}1a & a & a \\ & 3b & a \\ & & 3c\end{array}\alpha + \begin{array}{c|c|c|}1a & a & a \\ & 3b & a \\ & & 3c\end{array}\beta + \ldots\ldots + \begin{array}{c|c|c|}1\mathfrak{A} & \mathfrak{A} & \mathfrak{A} \\ & 3\mathfrak{B} & \mathfrak{A} \\ & & 3\mathfrak{C}\end{array}\varrho$$

$${}^{4}\Sigma = \begin{array}{c|c|c|c|}1A & A & A & A \\ & 4B & A & A \\ & & 4C & A \\ & & 2B & B \\ & & & 4D\end{array} = \begin{array}{c|c|c|c|}1a & a & a & a \\ & 4b & a & a \\ & & 4c & a \\ & & 2b & b \\ & & & 4d\end{array}\alpha + \begin{array}{c|c|c|c|}1a & a & a & a \\ & 4b & a & a \\ & & 4c & a \\ & & 2b & b \\ & & & 4b\end{array}\beta + \ldots + \begin{array}{c|c|c|c|}1\mathfrak{A} & \mathfrak{A} & \mathfrak{A} & \mathfrak{A} \\ & 4\mathfrak{B} & \mathfrak{A} & \mathfrak{A} \\ & & 4\mathfrak{C} & \mathfrak{A} \\ & & 2\mathfrak{B} & \mathfrak{B} \\ & & & 4\mathfrak{D}\end{array}\varrho$$

Ex formula *II.* vero sequitur, posito $n = 1;\ 2;\ 3;\ 4;$

$$
\begin{aligned}
{}^1\Sigma &= 1A = (1a)\alpha + (1\mathfrak{a})\beta + \ldots\ldots + (1\mathfrak{A})\varrho \\
{}^2\Sigma &= {}^1\Sigma A + 2B = (\underset{(a)}{i^1\mathfrak{J}a} + 2b)\alpha + (\underset{(\mathfrak{a})}{i^1\mathfrak{J}\mathfrak{a}} + 2\mathfrak{b})\beta + \ldots\ldots + (\underset{(\mathfrak{A})}{i^1\mathfrak{J}\mathfrak{A}} + 2\mathfrak{B})\varrho \\
{}^3\Sigma &= {}^2\Sigma A + {}^1\Sigma B + 3C = (\underset{(a,\ b,)}{i^2\mathfrak{J}a + i^1\mathfrak{J}b} + 3c)\alpha + (\underset{(\mathfrak{a},\ \mathfrak{b})}{i^2\mathfrak{J}\mathfrak{a} + i^1\mathfrak{J}\mathfrak{b}} + 3\mathfrak{c})\beta + \ldots + (\underset{(\mathfrak{A},\ \mathfrak{B})}{i^2\mathfrak{J}\mathfrak{A} + i^1\mathfrak{J}\mathfrak{B}} + 3\mathfrak{C})\varrho \\
{}^4\Sigma &= {}^3\Sigma A + {}^2\Sigma B + {}^1\Sigma C + 4D = (\underset{(a,\ b,\ c)}{i^3\mathfrak{J}a + i^2\mathfrak{J}b + i^1\mathfrak{J}c} + 4d)\alpha + \ldots + (\underset{(\mathfrak{A},\ \mathfrak{B},\ \mathfrak{C})}{i^3\mathfrak{J}\mathfrak{A} + i^2\mathfrak{J}\mathfrak{B} + i^1\mathfrak{J}\mathfrak{C}} + 4\mathfrak{D})\varrho
\end{aligned}
$$

§. XXIX.

Theorema.

Ex aequatione §. XXVII. *proposita*

$$[1-(ax+bx^2..)]^{\alpha}.[1-(\mathfrak{a}x+\mathfrak{b}x^2..)]^{\beta}..[1-(\mathfrak{A}x+\mathfrak{B}x^2..)]^{\varrho} = 1-(Ax+Bx^2+Cx^3..+Nx^n..)$$

sequitur:

$$N = \frac{1}{1}\mathfrak{a}^n A - \frac{1}{1.2}\mathfrak{b}^n B + \frac{1}{1.2.3}\mathfrak{c}^n C - \ldots + \frac{1}{1.2.3\ldots m}\mathfrak{m}^n \mathcal{M} + \ldots + \frac{1}{1.2..n}\mathfrak{n}^n \mathcal{N}$$
$$\left(\begin{array}{cccccc} \frac{{}^1\Sigma}{1}, & \frac{{}^2\Sigma}{2}, & \frac{{}^3\Sigma}{3}, & \frac{{}^4\Sigma}{4}, & \ldots\ldots & \frac{{}^n\Sigma}{n} \\ 1, & 2, & 3, & 4, & \ldots\ldots & n \end{array}\right)$$

Demonstratio.

Logarithmus naturalis factorum ad sinistram in aequatione est (§. XXVI.):

$$-\left(\frac{{}^1\Sigma}{1}x^1 + \frac{{}^2\Sigma}{2}x^2 + \frac{{}^3\Sigma}{3}x^3 + \frac{{}^4\Sigma}{4}x^4 + \ldots + \frac{{}^n\Sigma}{n}x^n + \ldots\right)$$

cui aequale sit $-\lambda$. Jam si h basin logarithmorum naturalium denotat, constat esse:

$$h^{-\lambda} = 1 - \frac{1}{1}\lambda + \frac{1}{1.2}\lambda^2 - \frac{1}{1.2.3}\lambda^3 + \ldots \mp \frac{1}{1.2.3\ldots n}\lambda^n \pm \ldots$$

ex qua aequatione, valore quantitatis $-\lambda$ restituto et dignitatibus eiusdem secundum §. VIII. expressis, sequitur:

$$h^{-\left[\frac{{}^1\Sigma}{1}x+\frac{{}^2\Sigma}{2}x^2\dots\right]} = 1-a^1Ax-\left.\begin{array}{l}\frac{1}{1}a^2A\\+\frac{1}{1.2}b^2B\end{array}\right|x^2-\left.\begin{array}{l}\frac{1}{1}a^3A\\+\frac{1}{1.2}b^3B\\-\frac{1}{1.2.3}c^3C\end{array}\right|x^3\begin{array}{l}-\dots\dots\\+\\-\\\end{array}\left.\begin{array}{l}-\frac{1}{1}a^nA\\+\frac{1}{1.2}b^nB\\-\frac{1}{1.2.3.}c^nC\\\mp\frac{1}{1.2..n}n^n\mathcal{N}\end{array}\right|x^n\dots \quad [Q$$

$$\underset{1,\ 2,\ 3,\ \dots\ n}{\left(\frac{{}^1\Sigma}{1},\frac{{}^2\Sigma}{2},\frac{{}^3\Sigma}{3},\dots\frac{{}^n\Sigma}{n}\dots\right)}$$

Est vero $1-[Ax+Bx^2+Cx^3+\dots+Nx^n\dots]=h^{Log.nat.[1-(ax+bx^2..)]^{\alpha}..[1-(\mathfrak{A}x+\mathfrak{B}x^2..)]^{\beta}}$

$$=h^{-\left[\frac{{}^1\Sigma x}{1}+\frac{{}^2\Sigma x^2}{2}+\frac{{}^3\Sigma x^3}{3}\dots+\frac{{}^n\Sigma x^n}{n}\dots\right]}$$

Itaque, ſerierum $1-[Ax+Bx^2+Cx^3+\dots+Nx^n\dots]$ et Q; comparatione inſtituta, reperitur:

$$N=\frac{1}{1}a^nA-\frac{1}{1.2}b^nB+\frac{1}{1.2.3}c^nC-\frac{1}{1.2.3.4.}b^nD+\dots\pm\frac{1}{1.2\dots m}m^n\mathcal{M}\mp\dots\pm\frac{1}{1.2\dots n}n^n\mathcal{N}$$

$$\underset{1,\ 2,\ 3,\ \dots\ n,}{\left(\frac{{}^1\Sigma}{1},\frac{{}^2\Sigma}{2},\frac{{}^3\Sigma}{3},\dots\frac{{}^n\Sigma}{n}\right)}$$

Sic eſt, poſito $n=1;\ 2;\ 3;\ 4;$

$$A=\frac{1}{1}\frac{{}^1\Sigma}{1}$$

$$B=\frac{1}{1}\frac{{}^1\Sigma}{2}-\frac{1}{1.2}\ \frac{{}^1\Sigma}{1}\frac{{}^1\Sigma}{1}$$

$$C=\frac{1}{1}\frac{{}^1\Sigma}{3}-\frac{1}{1.2}\ 2\frac{{}^1\Sigma}{1}\frac{{}^2\Sigma}{2}+\frac{1}{1.2.3.}\ \frac{{}^1\Sigma}{1}\frac{{}^1\Sigma}{1}\frac{{}^1\Sigma}{1},$$

$$D=\frac{1}{1}\frac{{}^4\Sigma}{4}-\frac{1}{1.2}\left[\begin{array}{l}2\frac{{}^1\Sigma}{1}\frac{{}^3\Sigma}{3}\\\frac{{}^2\Sigma}{2}\frac{{}^2\Sigma}{2}\end{array}\right]+\frac{1}{1.2.3.}3\frac{{}^1\Sigma}{1}\frac{{}^1\Sigma}{1}\frac{{}^2\Sigma}{2}-\frac{1}{1.2.3.4}\ \frac{{}^1\Sigma}{1}\frac{{}^1\Sigma}{1}\frac{{}^1\Sigma}{1}\frac{{}^1\Sigma}{1}$$

§. XXX.

§. XXX.

Problema.

Datis aequationibus numero r

$$
\begin{aligned}
{}^1s &= {}^1a\alpha + {}^1b\beta + {}^1c\gamma + \ldots\ldots + {}^1r\varrho \\
{}^2s &= {}^2a\alpha + {}^2b\beta + {}^2c\gamma + \ldots\ldots + {}^2r\varrho \\
{}^3s &= {}^3a\alpha + {}^3b\beta + {}^3c\gamma + \ldots\ldots + {}^3r\varrho \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
{}^rs &= {}^ra\alpha + {}^rb\beta + {}^rc\gamma + \ldots\ldots + {}^rr\varrho \ ^{*})
\end{aligned}
$$

valores quantitatum incognitarum $\alpha, \beta, \gamma, \ldots. \varrho$ eliminando quaerere.

Solutio.

Pars prior. Reperitur α regulis, quae sequuntur:

1) Complexio *abcd* .... *r* toties in columna verticali scribatur, quoties elementa, 1, 2, 3, 4 .... *r* permutari possunt (Cf. §. VII*).

2) In hac Columna (1) literis singulis Complexionis primae adscribantur, veluti exponentes, a laeva elementa singula primae permutationum a numeris 1, 2, 3 .... *r* oriundarum (prodit ${}^1a\,{}^2b\,{}^3c \ldots {}^rr$), et literis complexionis secundae elementa secundae permutationis, literis complexionis tertiae elementa tertiae permutationis, etc.

3) I. Complexionibus columnae (2), quae locis imparibus constitutae sunt, praefigantur vicissim signa + et — donec ${}^1a$ mutetur in ${}^2a$, deinde signa — et + donec ${}^2a$ mutetur in ${}^3a$, deinde iterum signa + et — donec ${}^3a$ mutetur in ${}^4a$, etc.
II. Complexionum paribus locis constitutarum cuique praefigatur contrarium signum complexionis proxime praecedentis.

4) Repetantur ea quae supra (1, 2, 3) praecepta fuerunt, sed tamen ita, ut in hac altera, quae prodit, columna ubique ponatur s loco a.

5) *Summa* complexionum cum suis signis columnae *posterioris*, per *summam* prioris divisa, valorem aequabit quantitatis α.

Pars

*) Hic literae $a, b, c, \ldots$ significant *coefficientes*, primum, secundum, tertium etc. literarum autem exponentes a laeva 1, 2, 3 .... *aequationes*, primam, secundam, tertiam etc. Sic verbi causa ${}^2c$ secundae aequationis tertium coefficientem denotat, et ${}^3a$ aequationis tertiae coefficientem primum.

Pars posterior. Eodem modo inveniuntur $\beta, \gamma, \delta, \ldots \varrho$, nisi quod s loco $b, c, d \ldots r$, respective ponatur (ut in 4. *Partis prioris* s pro a.).

Exempla.

1) Sit $r = 1$ et $\varrho = \alpha$. Ergo ${}^1s = {}^1a\alpha$ et

$$\alpha = {}^1s : {}^1a$$

2) Sit $r = 2$ et $\varrho = \beta$. Ergo

$$\begin{aligned} {}^1s &= {}^1a\alpha + {}^1b\beta \\ {}^2s &= {}^2a\alpha + {}^2b\beta \end{aligned} \quad \text{et}$$

$$\alpha = \begin{bmatrix} +{}^1s\,{}^2b \\ -{}^2s\,{}^1b \end{bmatrix} : \begin{bmatrix} +{}^1a\,{}^2b \\ -{}^2a\,{}^1b \end{bmatrix},$$

$$\beta = \begin{bmatrix} +{}^1a\,{}^2s \\ -{}^2a\,{}^1s \end{bmatrix} : \begin{bmatrix} +{}^1a\,{}^2b \\ -{}^2a\,{}^1b \end{bmatrix}$$

3) Sit $r = 3$ et $\varrho = \gamma$. Ergo

$$\begin{aligned} {}^1s &= {}^1a\alpha + {}^1b\beta + {}^1c\gamma \\ {}^2s &= {}^2a\alpha + {}^2b\beta + {}^2c\gamma \\ {}^3s &= {}^3a\alpha + {}^3b\beta + {}^3c\gamma \end{aligned} \quad \text{et}$$

$$\alpha = \begin{Bmatrix} +{}^1s\,{}^2b\,{}^3c \\ -{}^1s\,{}^3b\,{}^2c \\ -{}^2s\,{}^1b\,{}^3c \\ +{}^2s\,{}^3b\,{}^1c \\ +{}^3s\,{}^1b\,{}^2c \\ -{}^3s\,{}^2b\,{}^1c \end{Bmatrix} : \begin{Bmatrix} +{}^1a\,{}^2b\,{}^3c \\ -{}^1a\,{}^3b\,{}^2c \\ -{}^2a\,{}^1b\,{}^3c \\ +{}^2a\,{}^3b\,{}^1c \\ +{}^3a\,{}^1b\,{}^2c \\ -{}^3a\,{}^2b\,{}^1c \end{Bmatrix}, \quad \beta = \begin{Bmatrix} +{}^1a\,{}^2s\,{}^3c \\ -{}^1a\,{}^3s\,{}^2c \\ -{}^2a\,{}^1s\,{}^3c \\ +{}^2a\,{}^3s\,{}^1c \\ +{}^3a\,{}^1s\,{}^2c \\ -{}^3a\,{}^2s\,{}^1c \end{Bmatrix} : \begin{Bmatrix} +{}^1a\,{}^2b\,{}^3c \\ -{}^1a\,{}^3b\,{}^2c \\ -{}^2a\,{}^1b\,{}^3c \\ +{}^2a\,{}^3b\,{}^1c \\ +{}^3a\,{}^1b\,{}^2c \\ -{}^3a\,{}^2b\,{}^1c \end{Bmatrix}, \quad \gamma = \begin{Bmatrix} +{}^1a\,{}^2b\,{}^3s \\ -{}^1a\,{}^3b\,{}^2s \\ -{}^2a\,{}^1b\,{}^3s \\ +{}^2a\,{}^3b\,{}^1s \\ +{}^3a\,{}^1b\,{}^2s \\ -{}^3a\,{}^2b\,{}^1s \end{Bmatrix} : \begin{Bmatrix} +{}^1a\,{}^2b\,{}^3c \\ -{}^1a\,{}^3b\,{}^2c \\ -{}^2a\,{}^1b\,{}^3c \\ +{}^2a\,{}^3b\,{}^1c \\ +{}^3a\,{}^1b\,{}^2c \\ -{}^3a\,{}^2b\,{}^1c \end{Bmatrix}$$

Severa solutionis hujus demonstratio nititur in theoria *Variationum, omissis quidem repetitionibus*, id quod, alia scribendi occasione data, ostendam.

Scholion. Problematis solutio convenit cum regulis *Crameri* (*Introduction à l' Analyse des Courbes* p. 656. seqq. *Num. I. et II. De l' Evanouissement des Inconnues*). Easdem Hindenburgius, simul cum regulis de eadem re a Bezolto traditis (Bezout *Théorie générale des Equations algébriques*), amplissime pertractavit in praefatione libelli *Rüdigeriani*, cui titulus est: *Specimen analyticum de lineis curvis secundi ordinis*, ubi etiam p. XLVI — XLVIII. regula traditur datae complexionis permutationes reperiundi, qua h. l. (Sol. Pars prior 1, 2.) opus est.

§. XXXI.

Problema.

Propositae aequationis:

$$[1-(ax+bx^2..)]^\alpha[1-(\mathfrak{a}x+\mathfrak{b}x^2...)]^\beta..[1-(\mathfrak{A}x+\mathfrak{B}x^2)]^\varrho = 1-(\mathrm{A}x+\mathrm{B}x^2+\mathrm{C}x^3..+\mathrm{N}x^n..)$$

exponentes incognitos $\alpha, \beta, .. \varrho$ *per coefficientes datos*, $a, b, c \ldots$; $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}..$; $\mathfrak{A}, \mathfrak{B}, \mathfrak{C} \ldots$; et $\mathrm{A}, \mathrm{B}, \mathrm{C} \ldots$ exprimere.

Solu-

Solutio.

Secundum §. XXVIII. posito $n = 1, 2, 3, \ldots r$, est:

$$\begin{aligned}
\underset{(A)}{i^{1}J} &= \underset{(a)}{i^{1}J\alpha} + \underset{(\mathfrak{a})}{i^{1}J\beta} + \ldots\ldots + \underset{(\mathfrak{A})}{i^{1}J\varrho} \\
\underset{(A,B)}{i^{2}J} &= \underset{(a,b)}{i^{2}J\alpha} + \underset{(\mathfrak{a},\mathfrak{b})}{i^{2}J\beta} + \ldots\ldots + \underset{(\mathfrak{A},\mathfrak{B})}{i^{2}J\varrho} \\
\underset{(A,B,C)}{i^{3}J} &= \underset{(a,b,c)}{i^{3}J\alpha} + \underset{(\mathfrak{a},\mathfrak{b},\mathfrak{c})}{i^{3}J\beta} + \ldots\ldots + \underset{(\mathfrak{A},\mathfrak{B},\mathfrak{C})}{i^{3}J\varrho} \\
&\cdot \qquad \cdot \qquad \cdot \qquad \cdot \\
\underset{(A,B,\ldots R)}{i^{r}J} &= \underset{(a,b,\ldots r)}{i^{r}J\alpha} + \underset{(\mathfrak{a},\mathfrak{b},\ldots \mathfrak{r})}{i^{r}J\beta} + \ldots\ldots + \underset{(\mathfrak{A},\mathfrak{B},\ldots \mathfrak{R})}{i^{r}J\varrho}
\end{aligned}$$

Ex his aequationibus eruuntur (§. XXX.) valores literarum $\alpha, \beta \ldots \varrho$ quaesiti. Nam,

1) posito $r = 1$ et $\varrho = \alpha$, fit $\underset{(A)}{i^{1}J} = \underset{(a)}{i^{1}J}\alpha$. Ergo: $\alpha = + \underset{(A)}{i^{1}J} : \underset{(a)}{i^{1}J}$

2) posito $r = 2$ et $\varrho = \beta$, fit
$$\begin{aligned}
\underset{(A)}{i^{1}J} &= \underset{(a)}{i^{1}J\alpha} + \underset{(\mathfrak{a})}{i^{1}J\beta} \\
\underset{(A,B)}{i^{2}J} &= \underset{(a,b)}{i^{2}J\alpha} + \underset{(\mathfrak{a},\mathfrak{b})}{i^{2}J\beta}
\end{aligned}$$
Ergo:

$$\alpha = \underset{(A,B)\,(\mathfrak{a},\mathfrak{b})}{\begin{Bmatrix} +i^{1}J & i^{2}J \\ -i^{2}J & i^{1}J \end{Bmatrix}} : \underset{(a,b)\,(\mathfrak{a},\mathfrak{b})}{\begin{Bmatrix} +i^{1}J & i^{2}J \\ -i^{2}J & i^{1}J \end{Bmatrix}} \qquad \beta = \underset{(a,b)\,(A,B)}{\begin{Bmatrix} +i^{1}J & i^{2}J \\ -i^{2}J & i^{1}J \end{Bmatrix}} : \underset{(a,b)\,(\mathfrak{a},\mathfrak{b})}{\begin{Bmatrix} +i^{1}J & i^{2}J \\ -i^{2}J & i^{1}J \end{Bmatrix}}$$

ubi quilibet index pertinet ad signa involutoria seriei verticalis super ipsum positae.

Scholion. Dignum notari hoc problema, quippe methodorum vulgarium nulla, quod equidem sciam, ad illud solvendum sufficiat, ejusdem potius solutio videatur impossibilis, cum incognitae *plures* $\alpha, \beta, \gamma, \ldots \varrho$, *una* ex aequatione sint determinandae. *Analysis* igitur *Combinatoria* h. l. uti saepius, aliis methodis efficit majora.

Sed haec jam sufficiant, plura in additamentis.

ADDI-

ADDITAMENTA.

§ 1.

Summam libelli hactenus proposita complectuntur. Varia Corollaria et Exempla, quae adhuc reliqua sunt, nunc demum sequuntur, quod relationibus §. XXVIII. – XXXI. exhibitis interjecta, ipsarum ordinem turbassent.

§ 2.

Si in aequatione §. XXVII. exposita factorum multiplicandorum omnes sunt binomiorum potentiae *) veluti:

$$(1-ax)^{\alpha}(1-bx)^{\beta}(1-cx)^{\gamma}(1-dx)^{\delta}\ldots(1-rx)^{\varrho}=1-(\mathrm{A}x+\mathrm{B}x^2+\mathrm{C}x^3+\mathrm{D}x^4+\ldots+\mathrm{N}x^n+\ldots)$$

tunc est (§. XXV.):

$$^{n}\Sigma = a^n\alpha + b^n\beta + c^n\gamma + d^n\delta \ldots + r^n\varrho$$

et sequitur ex §. XXVIII. (mutato scilicet valore signi $^{n}\Sigma$)

$$I.\quad {}^{n}\Sigma = \underset{(\mathrm{A},\mathrm{B},\mathrm{C},\ldots\mathrm{N})}{i^n\mathrm{J}} = a^n\alpha + b^n\beta + c^n\gamma + d^n\delta + \ldots + r^n\varrho$$

$$II.\quad {}^{n}\Sigma = {}^{n-1}\Sigma\mathrm{A} + {}^{n-2}\Sigma\mathrm{B} + {}^{n-3}\Sigma\mathrm{C} + \ldots + {}^{n-m}\Sigma\mathrm{M} \ldots + {}^{1}\Sigma\mathrm{N} + n\mathrm{N} = a^n\alpha + b^n\beta + c^n\gamma \ldots + r^n\varrho$$

atque ex §. XXIX.

$$III.\quad \mathrm{N} = \frac{1}{1}\mathfrak{a}^n\mathrm{A} - \frac{1}{1.2}\mathfrak{b}^n\mathrm{B} + \frac{1}{1.2.3}\mathfrak{c}\,\mathrm{C} - \frac{1}{1.2.3.4}\mathfrak{d}^n\mathrm{D} + \ldots \pm \frac{1}{1.2\ldots m}\mathfrak{m}^n\mathcal{M} \mp \ldots \pm \frac{1}{1.2\ldots n}\mathfrak{n}^n\mathcal{N}$$

$$\left(\begin{matrix}\frac{{}^{1}\Sigma}{1}, & \frac{{}^{2}\Sigma}{2}, & \frac{{}^{3}\Sigma}{3}, & \frac{{}^{4}\Sigma}{4}, & \ldots & \frac{{}^{n}\Sigma}{n}\\ 1, & 2, & 3, & 4, & \ldots & n\end{matrix}\right)$$

§. 3.

*) Hanc aequationem tractarunt T. Simpson (*Philosophical Transactions Vol. XLVII.* 1751. p. 20.) et G. F. Tempelhofius, *Vir Strenuissimus*, (*Anfangsgründe der Analysis des Unendlichen* p. 301.)

§. 3.

PROBLEMA.

Summam reperire seriei infinitae:

$$1+\frac{1}{2^{2n}}+\frac{1}{3^{2n}}+\frac{1}{4^{2n}}+\frac{1}{5^{2n}}+\frac{1}{6^{2n}}+\frac{1}{7^{2n}}+\ldots\ldots,$$

in qua n esse potest numerus quilibet integer positivus.

SOLUTIO.

$$\frac{\sin z}{z}=\left(1-\frac{1.z^2}{1^2\pi^2}\right)\left(1-\frac{1.z^2}{2^2\pi^2}\right)\left(1-\frac{1.z^2}{3^2\pi^2}\right)\ldots=1-\left(\frac{z^2}{1.2.3}-\frac{z^4}{1.2.3.4.5}+\ldots\pm\frac{z^{2n}}{1.2\ldots(2n+1)}\mp\ldots\right)$$

Itaque secundum §. 2. formulam *I* *)

$$^{n}\Sigma=\frac{i^n J}{\left(\frac{1}{1.2.3},\ \frac{-1}{1.2.3.4.5},\ldots\frac{\mp 1}{1.2..(2n+1)}\right)}=\left(\frac{1}{1^{2n}}+\frac{1}{2^{2n}}+\frac{1}{3^{2n}}+\frac{1}{4^{2n}}+\ldots\right)\frac{1}{\pi^{2n}}$$

et secundum formulam *II*.

$$^{n}\Sigma=\frac{^{n-1}\Sigma}{1.2.3}-\frac{^{n-2}\Sigma}{1.2.3.4.5}+\frac{^{n-3}\Sigma}{1.2.3.4.5.6.7}-\ldots\pm\frac{^{n-m}\Sigma}{1.2\ldots(2m+1)}\mp\ldots\mp\frac{^{1}\Sigma}{1.2..(2n-1)}\pm\frac{n}{1.2..(2n+1)}$$

$$=\left(\frac{1}{1^{2n}}+\frac{1}{2^{2n}}+\frac{1}{3^{2n}}+\frac{1}{4^{2n}}+\ldots\right)\frac{1}{\pi^{2n}}.$$

COROLLARIUM.

Multiplicetur per $\frac{2}{2^{2n}}$ aequatio sequens:

*) Valores literarum in §. 2. hoc loco sunt:

$x=z^2;\quad \alpha=1,\quad \beta=1;\quad \gamma=1;\quad \delta=1;\ldots$

$a=\frac{1}{1^2\pi^2};\ b=\frac{1}{2^2\pi^2};\ c=\frac{1}{3^2\pi^2};\ d=\frac{1}{4^2\pi^2};\ldots$

$A=\frac{1}{1.2.3};\ B=\frac{-1}{1.2.3.4.5};\ C=\frac{1}{1.2.3.4.5.6.7};\ D=\frac{-1}{1.2.3.4.5.6.7.8.9};\ldots\ldots$

1 +

$$1+\frac{1}{2^{2n}}+\frac{1}{3^{2n}}+\frac{1}{4^{2n}}+\frac{1}{5^{2n}}+\ldots=\frac{[^n]\pi^{2n}}{\left(\frac{1}{1.2.3},\frac{-1}{1.2.3.4.5}\ldots\frac{\pm 1}{1.2\ldots(2n+1)}\right)}$$

hinc oritur:

$$\frac{2}{2^{2n}}+\frac{2}{4^{2n}}+\frac{2}{6^{2n}}+\frac{2}{8^{2n}}+\ldots=\frac{\frac{2}{2^{2n}}[^n]\pi^{2n}}{\left(\frac{1}{1.2.3},\frac{-1}{1.2.3.4.5}\ldots\frac{\pm 1}{1.2..(2n+1)}\right)}$$

qua ferie de priori detracta, provenit:

$$1-\frac{1}{2^{2n}}+\frac{1}{3^{2n}}-\frac{1}{4^{2n}}+\frac{1}{5^{2n}}-\ldots=\frac{\left(1-\frac{1}{2^{2n-1}}\right)[^n]\pi^{2n}}{\left(\frac{1}{1.2.3},\frac{-1}{1.2.3.4.5}\ldots\frac{\pm 1}{1.2..(2n+1)}\right)}$$

§. 4

PROBLEMA.

Summam reperire feriei infinitae:

$$1+\frac{1}{3^{2n}}+\frac{1}{5^{2n}}+\frac{1}{7^{2n}}+\frac{1}{9^{2n}}+\frac{1}{11^{2n}}+\frac{1}{13^{2n}}+\ldots$$

in qua n numerus esse potest quilibet integer positivus.

SOLUTIO.

$$Cos\, z=\left(1-\frac{2^2z^2}{1^2\pi^2}\right)\left(1-\frac{2^2z^2}{3^2\pi^2}\right)\left(1-\frac{2^2z^2}{5^2\pi^2}\right)\ldots=1-\left(\frac{z^2}{1.2}-\frac{z^4}{1.2.3.4}+\frac{z^6}{1.2.3.4.5.6}-\ldots\right)$$

Itaque fecundum §. 2. formulam *I.* *) $^n\Sigma=$

*) Valores literarum in §. 2. hoc loco funt: $x=\frac{2^2}{\pi^2}$ porro $\alpha=1$

$$ {}^{n}\Sigma = \frac{[^{n}]}{\left(\frac{1}{1.2}, \frac{-1}{1.2.3.4}, \dots \frac{\pm 1}{1.2..2n}\right)} = \left(1 + \frac{1}{3^{2n}} + \frac{1}{5^{2n}} + \frac{1}{7^{2n}} + \dots\right)\frac{2^{2n}}{\pi^{2n}} $$

et secundum formulam II.

$$ {}^{n}\Sigma = \frac{{}^{n-1}\Sigma}{1.2} - \frac{{}^{n-2}\Sigma}{1.2.3.4} + \frac{{}^{n-3}\Sigma}{1.2.3.4.5.6} - \dots \mp \frac{{}^{1}\Sigma}{1.2..(2n-2)} \pm \frac{n}{1.2..2n} = \left(1 + \frac{1}{3^{2n}} + \frac{1}{5^{2n}} + \frac{1}{7^{2n}} \dots\right)\frac{2^{2n}}{\pi^{2n}} $$

§. 5.

Problema.

Invenire summam seriei infinitae

$$ \frac{1}{1^{n}} \mp \frac{1}{3^{n}} + \frac{1}{5^{n}} \mp \frac{1}{7^{n}} + \frac{1}{9^{n}} \mp \frac{1}{11^{n}} + \frac{1}{13^{n}} \mp \dots, $$

ubi superiora signa valent, si n est numerus integer positivus impar, inferiora, si est par.

Solutio.

Secundum Eulerum (*Introductio in Analysin infinitorum Lib. I.* §. 171. et 175) est, posito $\frac{\pi x}{4} = z$:

$$ Cos\, z + Sin\, z \begin{cases} = \left(1 + \frac{4z}{1\pi}\right)\left(1 - \frac{4z}{3\pi}\right)\left(1 + \frac{4z}{5\pi}\right)\left(1 - \frac{4z}{7\pi}\right)\left(1 + \frac{4z}{9\pi}\right)\left(1 - \frac{4z}{11\pi}\right)\dots \\ = 1 - \left(-\frac{z}{1} + \frac{z^{2}}{1.2} + \frac{z^{3}}{1.2.3} - \frac{z^{4}}{1.2.3.4} - \dots \mp \frac{z^{2n-1}}{1.2(2n-1)} \pm \frac{z^{2n}}{1.2\dots 2n} \pm \dots\right) \end{cases} $$

Itaque

$$ \alpha = 1; \quad \beta = 1; \quad \gamma = 1; \quad \delta = 1; \dots $$

$$ a = \frac{2^{2}}{2^{2}\pi^{2}};\ b = \frac{2^{2}}{3^{2}\pi^{2}};\ c = \frac{2^{2}}{5^{2}\pi^{2}}\ d = \frac{2^{2}}{7^{2}\pi^{2}}; \dots $$

$$ A = \frac{1}{1.2};\ B = \frac{-1}{1.2.3.4};\ C = \frac{1}{1.2.3.4.5.6};\ D = \frac{1-}{1.2.3.4.5.6.7.8}; \dots $$

Itaque secundum §. 2. formulam I. *)

$$-{}^{2n-1}\Sigma = \frac{-\,[{}^{2n-1}]J}{\left(\frac{-1}{1}, \frac{1}{1.2}, \frac{1}{1.2.3}, \frac{-1}{1.2.3.4}, \dots \frac{\mp 1}{1.2..(2n-1)}\right)} = \left(\frac{1}{1^{2n-1}} - \frac{1}{3^{2n-1}} + \frac{1}{5^{2n-1}} - \frac{1}{7^{2n-1}} + \dots\right)\frac{4^{2n-1}}{\pi^{2n-1}}$$

$$+{}^{2n}\Sigma = \frac{+\,[{}^{2n}]J}{\left(\frac{-1}{1}, \frac{1}{1.2}, \frac{-1}{1.2.3}, \frac{-1}{1.2.3.4}, \dots \frac{\pm 1}{1.2.2n}\right)} = \left(\frac{1}{1^{2n}} + \frac{1}{3^{2n}} + \frac{1}{5^{2n}} + \frac{1}{7^{2n}} + \dots\right)\frac{4^{2n}}{\pi^{2n}}$$

et secundum formulam II.

$$-{}^{2n-1}\Sigma = -\frac{{}^{2n-2}\Sigma}{1} + \frac{{}^{2n-3}\Sigma}{1.2} + \frac{{}^{2n-4}\Sigma}{1.2.3} - \dots \pm \frac{{}^{1}\Sigma}{1.2..(2n-3)} = \left(\frac{1}{1^{2n-1}} - \frac{1}{3^{2n-1}} + \frac{1}{5^{2n-1}} - \dots\right)\frac{4^{2n-1}}{\pi^{2n-1}}$$

$$+{}^{2n}\Sigma = -\frac{{}^{2n-1}\Sigma}{1} + \frac{{}^{2n-2}\Sigma}{1.2} + \dots \pm \frac{{}^{3}\Sigma}{1.2..(2n-3)} \mp \frac{{}^{2}\Sigma}{1.2..(2n-2)} \mp \frac{{}^{1}\Sigma}{1.2..(2n-1)} = \left(\frac{1}{1^{2n}} + \frac{1}{3^{2n}} + \frac{1}{5^{2n}} \dots\right)\frac{4^{2n}}{\pi^{2n}}$$

in quibus formulis signa superiora valent, si n est numerus impar, si par, inferiora.

Corollarium.

Quia (§.3.)

$$1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \frac{1}{4^{2n}} + \frac{1}{5^{2n}} + \dots = \frac{[{}^{n}]J\,\pi^{2n}}{\left(\frac{1}{1.2.3}, \frac{-1}{1.2.3.4.5}, \frac{1}{1.2.3.4.5.6}, \dots\right)}$$

et

$$\frac{1}{2^{2n}} + \frac{1}{4^{2n}} + \frac{1}{6^{2n}} + \frac{1}{8^{2n}} + \dots = \frac{\frac{1}{2^{2n}}\,[{}^{n}]J\,\pi^{2n}}{\left(\frac{1}{1.2.3}, \frac{-1}{1.2.3.4.5}, \frac{1}{1.2.3.4.5.6}, \dots\right)}$$

prodit.

*) Pro hac scilicet aequatione in §. 2.

$x = z$; $\alpha = 1$; $\beta = 1$; $\gamma = 1$; $\delta = 1$; ...

$a = -\frac{4}{1\pi}$; $b = \frac{4}{3\pi}$; $c = -\frac{4}{5\pi}$; $d = \frac{4}{7\pi}$; ...

$A = \frac{-1}{1}$; $B = \frac{1}{1.2}$; $C = \frac{1}{1.2.3}$; $D = \frac{-1}{1.2.3.4}$; $E = \frac{-1}{1.2.3.4.5}$; $F = \frac{1}{1.2.3.4.5.6}$; ...

prodit, posteriore serie de priore detracta,

$$1 + \frac{1}{3^{2m}} + \frac{1}{5^{2m}} + \frac{1}{7^{2m}} + \dots = \left(1 - \frac{1}{2^{2m}}\right) {}^{n}J \, . \, \pi^{2m} \left(\frac{1}{1.2.3}, \frac{-1}{1.2.3.4.5}, \frac{1}{1.2.3.4.5.6.7}, \dots\right)$$

Ejusdem seriei summa inventa est §. 4 et 5, unde sequitur:

$$1 + \frac{1}{3^{2m}} + \frac{1}{5^{2m}} \dots = {}^{n}J\left(\frac{\pi}{2}\right)^{2m} \left(\frac{1}{1.2}, \frac{-1}{1.2.3.4}, \frac{1}{1.2.3.4.5.6}, \dots\right) = \left(2^{2m} - 1\right) {}^{n}J\left(\frac{\pi}{2}\right)^{2m} \left(\frac{1}{1.2.3}, \frac{-1}{1.2.3.4.5}, \frac{1}{1.2..7}, \dots\right) = \frac{1}{2^{2m}} {}^{n}J\left(\frac{\pi}{2}\right)^{2m} \left(\frac{-1}{1}, \frac{+1}{1.2}, \frac{+1}{1.2.3}, \frac{-1}{1.2.3.4}, \dots\right);$$

§. 6.

Serierum quarundam summae proponuntur independentes, quae terminorum recurrentium auxilio hucusque exhibita sunt.

In egregio opere (*Versuch einer neuen Summationsmethode. Berlin* 1788.) Pfaffius, *Vir Celeberrimus*, cum aliis, quas instituit, quaestionibus gravissimis, magnum formularum recurrentium numerum tradidit, quibus multarum serierum summae ex circuli rectificatione pendentes exprimuntur. Ex iisdem nonnullae h. l. deliguntur, quarum summae artissimo vinculo cum propositis (§. 3. 4. 5) conjunctae, earundem auxilio in formulas independentes transformantur.

Summam seriei infinitae, ubi m numerus integer positivus est,

$$1 + \frac{1}{2^{2m}} + \frac{1}{3^{2m}} + \frac{1}{4^{2m}} + \frac{1}{5^{2m}} + \dots + \frac{1}{n^{2m}} + \dots$$

Pfaffius in opere laudato ita exhibet;

$$\frac{\pi^2}{1.2.3} \Sigma \frac{1}{n^{2m-2}} - \frac{\pi^4}{1.2.3.4.5} \Sigma \frac{1}{n^{2m-4}} + \dots \mp \frac{\pi^{2m-2}}{1.2..(2m-1)} \Sigma \frac{1}{n^2} \pm \frac{\pi^{2m} m}{1.2.3\dots(2m+1)} \text{ *)}$$

ean-

*) Nempe signo Σ Pfaffius exprimit summam seriei infinitae, cujus terminus generalis ea est functio numeri n, quae huic signo adjicitur, sic $\Sigma\, N$ summa est omnium valorum functionis N, qui proveniunt, posito $n = 1, 2, 3, 4 \dots$ Signo $\Sigma \pm N$ idem notat seriei summam, in qua terminorum signa ($+ -$) alternantur (Pfaff. ibid. p. 4 **).

eandem (§. 3) reperimus esse

$$i^{m}]\pi^{2m}$$
$$\left(\frac{1}{1.2.3}, \frac{-1}{1.2.3.4.5}, \frac{1}{1.2..7}, \ldots\right)$$

Pfaffiana formula recurrit, nostra vero est independens. Haec praemittenda erant, ut sequentia intelligantur, ubi Pf. cum numero adposito *Pfaffiani* operis paginam notat.

I. Summa seriei infinitae $Sin\ \varphi + \frac{Sin\ 2\varphi}{2^{2m-1}} + \frac{Sin\ 3\varphi}{3^{2m-1}} + \frac{Sin\ 4\varphi}{4^{2m-1}} + \ldots\ldots$

$$= \varphi\Sigma\frac{1}{2^{2m-2}} - \frac{\varphi^3}{1.2.3}\Sigma\frac{1}{2^{2m-4}} + \ldots \mp \frac{\varphi^{2m-3}}{1.2..(2m-3)}\Sigma\frac{1}{n^2} \pm \frac{\varphi^{2m-2}}{1.2..(2m-2)}\frac{\pi}{2} \mp \frac{\varphi^{2m-1}}{1.2..(2m-1)}\frac{1}{2} \qquad \text{(Pf. 10.}$$

$$= \varphi i^{m-1}]\pi^{2m-2} - \frac{\varphi^3}{1.2.3}i^{m-2}]\pi^{2m-4} + \ldots \mp \frac{\varphi^{2m-3}}{1.2..(2m-3)}i^1]\pi^2 \pm \frac{\varphi^{2m-2}}{1.2..(2m-2)}\frac{\pi}{2} \mp \frac{\varphi^{2m-1}}{1.2..(2m-1)}\frac{1}{2} \qquad \text{(§. 3.}$$
$$\left(\frac{1}{1.2.3}, \frac{-1}{1.2.3.4.5}, \frac{1}{1.2.3.4.5.6.7}, \frac{-1}{1.2.3\ldots9}, \ldots\right)$$

II. Summa seriei infinitae $Sin\ \varphi - \frac{Sin\ 2\varphi}{2^{2m-1}} + \frac{Sin\ 3\varphi}{3^{2m-1}} - \frac{Sin\ 4\varphi}{4^{2m-1}} + \ldots$

$$= \varphi\Sigma\pm\frac{1}{n^{2m-2}} - \frac{\varphi^3}{1.2.3}\Sigma\pm\frac{1}{n^{2m-4}} + \ldots \mp \frac{\varphi^{2m-3}}{1.2..(2m-3)}\Sigma\pm\frac{1}{n^2} \pm \frac{\varphi^{2m-1}}{1.2..(2m-1)}\frac{1}{2} \qquad \text{(Pf. 8.}$$

$$= \varphi\left(1-\frac{1}{2^{2m-3}}\right)i^{m-1}]\pi^{2m-2} - \frac{\varphi^3}{1.2.3}\left(1-\frac{1}{2^{2m-5}}\right)i^{m-2}]\pi^{2m-4} \ldots \mp \frac{\varphi^{2m-3}}{1.2..(2m-3)}\left(1-\frac{1}{2}\right)i^1]\pi^2 \pm \frac{\varphi^{2m-1}}{1.2..(2m-1)}\frac{1}{2} \qquad \text{(§.3.C.}$$
$$\left(\frac{1}{1.2.3}, \frac{-1}{1.2.3.4.5}, \frac{1}{1.2.3.4.5.6.7}, \ldots\ldots\right)$$

III. Summa seriei infinitae $Cos\ \varphi + \frac{Cos\ 2\varphi}{2^{2m}} + \frac{Cos\ 3\varphi}{3^{2m}} + \frac{Cos\ 4\varphi}{4^{2m}} + \ldots\ldots$

$$= \Sigma\frac{1}{n^{2m}} - \frac{\varphi^2}{1.2}\Sigma\frac{1}{n^{2m-2}} + \ldots \pm \frac{\varphi^{2m-2}}{1.2.(2m-2)}\Sigma\frac{1}{n^2} \mp \frac{\varphi^{2m-1}}{1.2..(2m-1)}\frac{1}{2} \pm \frac{1}{2}\frac{\varphi^{2m}}{1.2..2m} \qquad \text{(Pf. 12.}$$

$$= i^{m}]\pi^{2m} - \frac{\varphi^2}{1.2}i^{m-1}]\pi^{2m-2} + \ldots \pm \frac{\varphi^{2m-2}}{1.2..(2m-2)}i^1]\pi^2 \mp \frac{\varphi^{2m-1}}{1.2..(2m-1)}\cdot\frac{1}{2} \pm \frac{1}{2}\frac{\varphi^{2m}}{1.2..2m} \qquad \text{(§. 3.}$$
$$\left(\frac{1}{1.2.3}, \frac{-1}{1.2.3.4.5}, \frac{1}{1.2.3.4.5.6.7}, \ldots\right)$$

IV. Sum-

IV. Summa feriei infinitae $Cos\ \varphi - \frac{Cos\ 2\varphi}{2^{2m}} + \frac{Cos\ 3\varphi}{3^{2m}} - \frac{Cos\ 4\varphi}{4^{2m}} + \dots$:

$$= \Sigma \pm \frac{1}{n^{2m}} - \frac{\varphi}{1.2}\Sigma \pm \frac{1}{n^{2m-2}} + \dots \pm \frac{\varphi^{2m-2}}{1.2..(2m-2)}\ \Sigma \pm \frac{1}{n^2} \mp \frac{1}{2}\frac{\varphi^{2m}}{1.2..2m} \qquad \text{(Pf. 12.}$$

$$= \left(1 - \frac{1}{2^{2m-1}}\right)\mathrm{i}^m\mathrm{J}\pi^{2m} - \frac{\varphi^2}{1.2}\left(1 - \frac{1}{2^{2m-3}}\right)\mathrm{i}^{m-1}\mathrm{J}\pi^{2m-2} \dots \pm \frac{\varphi^{2m-2}}{1.2..(2m-2)}\left(1 - \frac{1}{2}\right)\mathrm{i}^1\mathrm{J}\pi^2 \mp \frac{1}{2}\frac{\varphi^2}{1.2..2m} \text{ (§. 3. C.}$$

$$\left(\frac{1}{1.2.3}, \frac{-1}{1.2.3.4.5}, \frac{1}{1.2.3.4.5.6.7}, \dots\right)$$

§. 7.

PROBLEMA.

Summam reperire feriei infinitae

$$Cos\ \varphi - \frac{Cos\ 3\varphi}{3^{2m-1}} + \frac{Cos\ 5\varphi}{5^{2m-1}} - \frac{Cos\ 7\varphi}{7^{2m-1}} + \dots\dots$$

in qua m numerus est integer positivus.

SOLUTIO.

Pfaffiana methodo Cofinus arcuum φ, 3φ, 5φ, ... folvantur in feries infinitas, eae multiplicentur refpective per 1; $-\frac{1}{3^{2m-1}}$; $+\frac{1}{5^{2m-1}}$; ... deinde factores numeratori et denominatori fingulorum terminorum communes extinguantur, termini ipfi fecundum potentias arcus φ disponantur, et coefficientes dignitatum arcus φ in fummas colligantur prodibit auxilio §. 5. fumma feriei propofitae

$$= -\mathrm{i}^{2m-1}\mathrm{J}\frac{\pi^{2m-1}}{4^{2m-1}} + \mathrm{i}^{2m-3}\mathrm{J}\frac{\pi^{2m-3}}{4^{2m-3}}\frac{\varphi^2}{1.2} - \dots \mp \mathrm{i}^3\mathrm{J}\frac{\pi^3}{4^3}\frac{\varphi^{2m-4}}{1.2..(2m-4)} \pm \mathrm{i}^1\mathrm{J}\frac{\pi^1}{4^1}\frac{\varphi^{2m-2}}{1.2..(2m-2)}$$

$$\left(\frac{-1}{1}, \frac{1}{1.2}, \frac{1}{1.2.3}, \frac{-1}{1.2.3.4}, \frac{-1}{1.2.3.4.5}, \dots\right)$$

(Conferatur PFAFFIUS l. c. X. 3. p. 39. feq.)

§. 8.

§. 8.

PROBLEMA.

Summam reperire seriei infinitae, in qua m numerus est integer positivus:

$$\text{Sin}\,\varphi - \frac{\text{Sin}\,3\,\varphi}{3^{2m}} + \frac{\text{Sin}\,5\,\varphi}{5^{2m}} - \frac{\text{Sin}\,7\,\varphi}{7^{2m}} + \dots$$

SOLUTIO.

Sinus solvantur in series infinitas, caeteraque agantur uti §. 7. prodit summa quaesita:

$$= -[^{2m-1}]\frac{\pi^{2m-1}}{4^{2m-1}}\varphi + [^{2m-3}]\frac{\pi^{2m-3}}{4^{2m-3}}\frac{\varphi^3}{1.2.3} - \dots \pm [^3]\frac{\pi^3}{4^3}\frac{\varphi^{2m-3}}{1.2..(2m-3)} \mp [^1]\frac{\pi^1}{4^1}\frac{\varphi^{2m-1}}{1.2..(2m-1)}$$

$$\left(\frac{-1}{1}, \frac{1}{1.2}, \frac{1}{1.2.3.}, \frac{-1}{1.2.3.4.}, \dots\right)$$

(Conferatur PFAFFIUS l. c. X. 5. p. 41.)

§. 9.

Eadem methodo, quae §. 7. et 8 adhibita est, sed auxilio Corollarii §. 3. eruuntur summae serierum sequentium, in quibus m numerus est integer positivus:

$$1)\ \frac{\text{Sin}\,2\,\varphi}{2^{2m+1}} - \frac{\text{Sin}\,4\,\varphi}{4^{2m+1}} + \frac{\text{Sin}\,6\,\varphi}{6^{2m+1}} - \dots = \left(1 - \frac{1}{2^{2m-1}}\right)[^m]\frac{\pi^{2m}}{2^{2m}}\varphi - \left(1 - \frac{1}{2^{2m-3}}\right)[^{m-1}]\frac{\pi^{2m-2}}{2^{2m-2}}\frac{\varphi^3}{1.2.3} + \dots$$

$$\dots \pm \left(1 - \frac{1}{2}\right)[^1]\frac{\pi^2}{2^2}\frac{\varphi^{2m-1}}{1.2..(2m-1)} \mp \frac{1}{2}\frac{\varphi^{2m+1}}{1.2..(2m+1)}$$

$$2)\ \frac{\text{Cos}\,2\,\varphi}{2^{2m}} - \frac{\text{Cos}\,4\,\varphi}{4^{2m}} + \frac{\text{Cos}\,6\,\varphi}{6^{2m}} - \dots = \left(1 - \frac{1}{2^{2m-1}}\right)[^m]\frac{\pi^{2m}}{2^{2m}} - \left(1 - \frac{1}{2^{2m-3}}\right)[^{m-1}]\frac{\pi^{2m-2}}{2^{2m-2}}\frac{\varphi^2}{1.2} + \dots$$

$$\dots \pm \left(1 - \frac{1}{2}\right)[^1]\frac{\pi^2}{2^2}\frac{\varphi^{2m-2}}{1.2..(2m-2)} \mp \frac{\varphi^{2m}}{1.2..2m}$$

ubi ad indicem $\left(\frac{1}{1.2.3}, \frac{-1}{1.2.3.4.5}, \frac{1}{1.2.3.4.5.6.7}, \dots\right)$ signa involutoria referuntur. Est enim secundum EULERUM (*Instit. Calc. Diff. P. II.* §. 185.)

$$1 - 2^{2m} + 3^{2m} - 4^{2m} + 5^{2m} - \dots = 0$$

Hoc itaque loco serierum trigonometricarum summis, sublato omni terminorum recursu, campus patet latissimus.

§. 10.

PROBLEMA.

Producti ex factoribus numero infinitis, secundum potentias variabilis x ordinati,

$$\left(1-\frac{x}{b}\right)^{a}\left(1-\frac{x}{b^2}\right)^{a}\left(1-\frac{x}{b^3}\right)^{a}\ldots = 1-(Ax+Bx^2+Cx^3+\ldots+Nx^n+\ldots)$$

invenire coefficientem quemlibet a prioribus independenter.

SOLUTIO.

Est hoc loco secundum §. 2. et formulam III.

$$\frac{{}^{n}\Sigma}{n}=\frac{1}{n}\left(\frac{a}{b^n}+\frac{a}{b^{2n}}+\frac{a}{b^{3n}}+\frac{a}{b^{4n}}+\ldots\right)=\frac{a}{n(b^n-1)}$$

atque

$$N=\frac{1}{1}a^nA-\frac{1}{1.2}b^nB+\frac{1}{1.2.3}c^nC-\ldots\pm\frac{1}{1.2\ldots m}m^n\mathcal{M}\ldots\pm\frac{1}{1.2\ldots n}n^n\mathcal{N}$$

$$\left(\underset{1,}{\frac{a}{1(b^1-1)}},\ \underset{2,}{\frac{a}{2(b^2-1)}},\ \underset{3,}{\frac{a}{3(b^3-1)}}\ \ldots\ \underset{n}{\frac{a}{n(b^n-1)}}\right)$$

unde prodit, posito $n=1, 2, 3, \ldots$

$$A=\frac{a}{b-1};\quad B=\frac{a}{2(b^2-1)}-\frac{a^2}{2(b-1)^2};\quad C=\frac{a}{3(b^3-1)}-\frac{a^2}{2(b-1)(b^2-1)}+\frac{a^3}{2.3.(b-1)^3}$$

§. 11.

Quodsi in aequatione §. 2. substituatur

$$\begin{Bmatrix}\text{pro} & \alpha, & \beta, & \gamma, & \delta, & \varepsilon, & \zeta,\ldots\\ \text{respective} & +\alpha, & -\alpha, & +\beta, & -\beta, & +\gamma, & -\gamma,\ldots\end{Bmatrix}\text{ et }\begin{Bmatrix}\text{pro} & a, & b, & c, & d, & e, & f,\ldots\\ \text{respective} & +a, & -a, & -b, & +b, & +c, & -c,\ldots\end{Bmatrix}\text{ oritur}$$

$$(1-ax)^{\alpha}(1+ax)^{-\alpha}(1+bx)^{+\beta}(-bx)^{-\beta}(1-cx)^{+\gamma}(1+cx)^{-\gamma}\ldots=1-(Ax+Bx^2+Cx^3\ldots+Nx^n\ldots)$$

ubi numerus factorum semper est par, sive sit infinitus, sive finitus, atque fit:

$${}^{2m-1}\Sigma=2\,[a^{2m-1}\alpha-b^{2m-1}\beta+c^{2m-1}\gamma-d^{2m-1}\delta+\ldots]$$
$${}^{2m}\Sigma=0$$

Hinc

Hinc provenit:

$$\text{I.}\quad {}^{2m-1}\Sigma = {}^{2m-3}\Sigma_B + {}^{2m-5}\Sigma_D + \ldots + {}^{1}\overset{m-2}{\Sigma}_M + (2m-1)\overset{m-1}{M} = 2\,[a^{2m-1}\alpha - b^{2m-1}\beta + c^{2m-1}\gamma - \ldots]$$

$$\text{II.}\quad {}^{2m}\Sigma = {}^{2m-1}\Sigma_A + {}^{2m-3}\Sigma_C + \ldots + {}^{3}\overset{m-2}{\Sigma}_M + {}^{1}\overset{m-1}{\Sigma}_M + 2m\overset{m}{M} = 0$$

$$\text{III.}\quad N = \pm\left(\frac{1}{1.2..n}\mathfrak{n}^n\mathcal{N} + \frac{1}{1.2..(n-2)}\mathfrak{n}^n\overset{-2}{\mathcal{N}} + \frac{1}{1.2..(n-4)}\mathfrak{n}^n\overset{-4}{\mathcal{N}} + \ldots\ldots + \left\{\begin{matrix}\frac{1}{1}\,a^nA\\ \frac{1}{1.2}\,b^nB\end{matrix}\right.\right).\ *)$$

$$\begin{pmatrix}\frac{{}^1\Sigma}{1}, & 0, & \frac{{}^3\Sigma}{3}, & 0, & \frac{{}^5\Sigma}{5}, & 0, & \frac{{}^7\Sigma}{7}, & \ldots\ldots\\ 1, & 2, & 3, & 4, & 5, & 6, & 7, & \ldots\ldots\end{pmatrix}$$

in qua tertia formula $\left\{\begin{array}{l}\text{superiora signa } + \text{ et } \frac{1}{1}a^nA \text{ valent, si } n \text{ numerus est impar,}\\ \text{inferiora } \cdot - \text{ et } \frac{1}{1.2}b^nB, \text{ si } n \text{ est par.}\end{array}\right.$

§. 12.

P R O B L E M A.

Invenire valores secundum potentias quantitatis x digestos productorum, quae sequuntur:

$$1)\ \left(\frac{1-x}{1+x}\right)^{Sin\,\varphi}\left(\frac{1+\frac{1}{2}x}{1-\frac{1}{2}x}\right)^{Sin\,2\varphi}\left(\frac{1-\frac{1}{3}x}{1+\frac{1}{3}x}\right)^{Sin\,3\varphi}\left(\frac{1+\frac{1}{4}x}{1-\frac{1}{4}x}\right)^{Sin\,4\varphi}\ldots = 1-(Ax+Bx^2+Cx^3\ldots+Nx^n\ldots)$$

$$2)\ \left(\frac{1-x}{1+x}\right)^{Cos\,\varphi}\left(\frac{1+\frac{1}{3}x}{1-\frac{1}{3}x}\right)^{Cos\,3\varphi}\left(\frac{1-\frac{1}{5}x}{1+\frac{1}{5}x}\right)^{Cos\,5\varphi}\left(\frac{1+\frac{1}{7}x}{1-\frac{1}{7}x}\right)^{Cos\,7\varphi}\ldots = 1-(\mathfrak{A}x+\mathfrak{B}x^2+\mathfrak{C}x^3\ldots+\mathfrak{N}x^n\ldots)$$

S O L U T I O.

Secundum §. 11. **) est in producto 1)

$$ {}^{2m-1}\Sigma = 2\,\left[Sin\,\varphi - \frac{Sin\,2\,\varphi}{2^{2m-1}} + \frac{Sin\,3\,\varphi}{3^{2m-1}} - \frac{Sin\,4\,\varphi}{4^{2m-1}} + \ldots\ldots\right]$$

in

*) Scripsi h. l. $\mathfrak{n}^n\overset{-2}{\mathcal{N}}$ pro $\overset{-2}{\mathfrak{n}}{}^n\overset{-2}{\mathcal{N}}$ etc. quod hoc compendio nulla obscuritas nascitur.

**) Est scilicet in §. 11. intuitu prioris producti

$\alpha = Sin\,\varphi;\ \beta = Sin\,2\,\varphi;\ \gamma = Sin\,3\,\varphi;\ldots\ a = 1;\ b = \frac{1}{2};\ c = \frac{1}{3};\ d = \frac{1}{4};\ldots$

intuitu posterioris

$\alpha = Cos\,\varphi;\ \beta = Cos\,3\,\varphi;\ \gamma = Cos\,5\,\varphi;\ldots\ a = 1;\ b = \frac{1}{3};\ c = \frac{1}{5};\ d = \frac{1}{7};\ldots$

in producto 2)

$$ {}^{2m-1}\Sigma = 2\left[\mathit{Cos}\,\varphi - \frac{\mathit{Cos}\,3\,\varphi}{3^{2m-1}} + \frac{\mathit{Cos}\,5\,\varphi}{5^{2m-1}} - \frac{\mathit{Cos}\,7\,\varphi}{7^{2m-1}} + \dots\dots\right] $$

cum vero harum ſerierum ſummae cognitae ſint (§. 6. II. et §. 7.), ſimul utriusque ſeriei coefficiens quilibet independenter erui poteſt. Invenitur:

$$ \mathrm{A} = \varphi;\quad \mathrm{B} = -\frac{\varphi^2}{1.2};\quad \mathrm{C} = \frac{\varphi^3}{1.2.3}\left(\frac{2\,\varphi^2 + \pi^2}{3}\right);\dots $$

$$ \mathfrak{A} = \frac{\pi}{2};\quad \mathfrak{B} = -\frac{\pi^2}{8};\quad \mathfrak{C} = \frac{\pi}{4}\left(\frac{\pi^2}{4^2} + \frac{\varphi^2}{2}\right);\dots $$

§. 13.

Ex aequatione

$$ (1-ax)(1-bx)(1-cx)\dots(1-rx) = 1 - (\mathrm{A}x + \mathrm{B}x^2 + \mathrm{C}x^3 + \dots + \mathrm{N}x^n \dots + \mathrm{R}x^r) $$

ſequitur ſecundum §. 2.

$$ \text{I.}\quad {}^n\Sigma = \underset{(\mathrm{A},\,\mathrm{B},\,\mathrm{C},\,\dots\,\mathrm{N})}{f^n J} = a^n + b^n + c^n + \dots + r^n. $$

$$ \text{II.}\quad {}^n\Sigma = {}^{n-1}\Sigma\mathrm{A} + {}^{n-2}\Sigma\mathrm{B} + \dots + {}^{1}\Sigma\mathrm{N} + n\mathrm{N} = a^n + b^n + c^n + \dots + r^n $$

$$ \text{III.}\quad \mathrm{N} = \frac{1}{1}\,a^n\mathrm{A} - \frac{1}{1.2}\,b^n\mathrm{B} + \frac{1}{1.2.3}\,c^n\mathrm{C} - \dots \pm \frac{1}{1.2..m}\,m^n\mathcal{M} \mp \dots \pm \frac{1}{1.2..n}\,n^n\mathcal{N} $$

$$ \left(\frac{{}^1\Sigma}{1}, \frac{{}^2\Sigma}{2}, \frac{{}^3\Sigma}{3}, \dots \frac{{}^n\Sigma}{n}\right) $$

Quodſi ponatur $1 - (\mathrm{A}x + \mathrm{B}x^2 + \dots + \mathrm{R}x^r) = 0$, ſimul $a,\ b,\ c, \dots r$; ſunt radices aequationis

$$ y^r - (\mathrm{A}y^{r-1} + \mathrm{B}y^{r-2} + \mathrm{C}y^{r-3} + \dots + \mathrm{N}y^{r-n} + \dots + \mathrm{R}) = 0 $$

in qua $y = \frac{1}{x}$ (*) et formularum propoſitarum prima et ſecunda exhibent ſummam radicum aequa-

*) Ex aequatione $(1-ax)(1-bx)\dots(1-rx) = 1-(\mathrm{A}x + \mathrm{B}x^2 \dots + \mathrm{R}x^r) = 0$ ſequitur,

poſito $x = \frac{1}{y}$; $\quad (1-\frac{a}{y})(1-\frac{b}{y})\dots(1-\frac{r}{y}) = 1-(\frac{\mathrm{A}}{y} + \frac{\mathrm{B}}{y^2} + \dots + \frac{\mathrm{R}}{y^r}) = 0$

ſive $\quad (\frac{y-a}{y})(\frac{y-b}{y})\dots(\frac{y-r}{y}) = 1-(\frac{\mathrm{A}}{y} + \frac{\mathrm{B}}{y^2} + \dots + \frac{\mathrm{R}}{y^r}) = 0$

et ſi haec aequatio multiplicetur per y^r,

$$ (y-a)(y-b)\dots(y-r) = y^r - [\mathrm{A}y^{r-1} + \mathrm{B}y^{r-2} \dots + \mathrm{R}\,y^0] = 0 $$

aequationis hujus ad potentiam *n*tam elevatarum; altera quidem independenter, altera vero, NEWTONI, BAERMANNI, KAESTNERI, EULERI, TEMPELHOFII, aliorumque Analystarum exemplo *), insertis praecedentibus radicum potentiis.

De tertia formula unum moneam: Exhibet ea valorem coefficientis N, sed constat, eundem aequare *n*tam classem omnium complexionum rite ordinatarum indicis, (*a*, *b*, *c*, ... *r*), in quibus singulis nullum elementum bis vel saepius occurrit, (KAESTNERI *Analysis endl. Grössen* §. 224.) i. e. *n*tam classem *Combinationum simpliciter, indicis* (*a*, *b*, *c*, *d*, ... *r*) *et omissis quidem repetitionibus* **), quam HINDENBURGIUS, *Vir Celeberrimus*, hoc notat signo:

$$\underset{(a,\, b,\, c,\, \ldots r)}{\overset{n}{\mathcal{N}}}$$

Itaque, duplici valore coefficientis N invento, haec prodit relatio:

$$\text{N} = \underset{(a,b,c,\ldots r)}{{}^{n}\mathcal{N}} = \frac{1}{1}\,\mathfrak{a}^{n}\mathfrak{A} - \frac{1}{1.2}\,\mathfrak{b}^{n}\mathfrak{B} + \ldots \pm \frac{1}{1.2\ldots m}\,\mathfrak{m}^{n}\mathcal{M} \mp \ldots \pm \frac{1}{1.2\ldots n}\,\mathfrak{n}^{n}\mathcal{N}.$$

$$\left(\frac{{}^{1}\Sigma}{1},\ \frac{{}^{2}\Sigma}{2},\ \frac{{}^{3}\Sigma}{3}, \ldots \frac{{}^{n}\Sigma}{n}\right) \qquad \text{§. 14.}$$

*) Theorema, quod formula ${}^{n}\Sigma = {}^{n-1}\Sigma\text{A} + {}^{n-2}\Sigma\text{B} + \ldots \pm {}^{1}\Sigma\text{N} \mp n\text{N}$ exprimitur, NEWTONUS (*Arithmetica Universalis in fine capitis de transmutationibus aequationum p.* 192. *Editionis s' Gravesandinae*) proposuit, sed nullam ejus demonstrationem. KAESTNERUS, *Vir Illustris*, illud demonstravit, afferens simul alia, quae pertinent ad hoc theorema (*Analysis endlicher Grössen* §. 751). *Eulerianas* hujus theorematis demonstrationes, MICHELSEN, *Vir Celeberrimus*, in additamentis suis ad EULERI *Introductionem in Analysin infinitorum* collegit (*Zusätze zum 10ten Capitel des 1sten Buchs*).

**) In tabula adposita Combinationum simpliciter, indicis (*a*, *b*, *c*, *d*, *e*) et omissis quidem repetitionibus, duarum vicinarum classium posterior ex priori oritur, si quaelibet prioris classis complexio ante indicis elementa, novissimum ipsius elementum insequentia, successive ponitur, atque ita complexiones ordinantur, ut quae in idem desinunt elementum, eadem in serie verticali collocentur.

$$\begin{array}{ccccc|l}
a & b & c & d & e) & = \text{A}' \\
\hline
 & ab & ac & ad & ae & \\
 & & bc & bd & be & = \text{B}' \\
 & & & cd & ce & \\
 & & & & de & \\
\hline
 & & abc & abd & abe & \\
 & & & acd & ace & \\
 & & & & ade & = \text{C}' \\
 & & & bcd & bce & \\
 & & & & bde & \\
 & & & & cde & \\
\hline
 & & & abcd & abce & \\
 & & & & abde & = \text{D}' \\
 & & & & acde & \\
 & & & & bcde & \\
\hline
 & & & & abcde & = \underset{(a,b,c,d,e)}{\text{E}'}
\end{array}$$

§. 14.

PROBLEMA.

Data aequatione

$$[1-(1x+1x^2)]^\alpha[1-(2x+2x^2)]^\beta = 1-(3x+4x^2+cx^3+dx^4+\dots)$$

reperire exponentes α, β *et coefficientes* C, D, E, ...

SOLUTIO.

Pars prior. Est

$$\begin{array}{lll} \{^1] = 1 & \{^1] = 2 & \{^1] = 3 \\ \{^2] = 3 & \{^2] = 8 & \{^2] = 17 \\ \{^3] = 4 & \{^3] = 20 & \{^3] = 63+3c \\ \{^4] = 7 & \{^4] = 56 & \{^4] = 257+12c+4d \\ (1,1) & (2,2) & (3,4,c,d) \end{array}$$

Secundum §. XXXI. reperitur:

$$\alpha = \underset{(3,4)\,(2,2)}{\left\{\begin{matrix}+\{^1] & \{^2] \\ -\{^2] & \{^1]\end{matrix}\right\}} : \underset{(1,1)\,(2,2)}{\left\{\begin{matrix}+\{^1] & \{^2] \\ -\{^2] & \{^1]\end{matrix}\right\}} = -5; \quad \beta = \underset{(1,1)\,(3,4)}{\left\{\begin{matrix}+\{^1] & \{^2] \\ -\{^2] & \{^1]\end{matrix}\right\}} : \underset{(1,1)\,(2,2)}{\left\{\begin{matrix}+\{^1] & \{^2] \\ -\{^2] & \{^1]\end{matrix}\right\}} = 4$$

Pars posterior. Secundum §. XXVIII.

$$\underset{(3,4,c)}{\{^3]} = -\underset{(1,1)}{\{^3]}5 + \underset{(2,2)}{\{^3]}4; \quad \text{hoc est:} \quad 63+3c = -4.5+20.4$$

ex qua aequatione eruitur $C = -1$. Secundum eundem §.

$$\underset{(3,4,c,d)}{\{^4]} = -\underset{(1,1)}{\{^4]}5 + \underset{(2,2)}{\{^4]}4 \quad \text{hoc est:} \quad 257+12c+4d = -7.5+56.4$$

unde sequitur: $D = -14$.

Eadem ratione valor reliquorum coefficientium E, F, G... successive invenitur.

His licet paucis exemplis relationum supra §. XXVIII. seqq. propositarum usus apparebit uberrimus.

EX OFFICINA KRAMERIANA.

Pag.	*Lin.*	*Errata.*	*Corrige*
1	4	adhibitae	adhibitae novitatem
3	12	(§. Def. 1)	(§. II. Def. 1)
4	4	113444	11344.
—	8	III.	II.
7	6 a fine	$123\ldots\nu$	$1.2.3\ldots\nu$
8	19	*5ccdd*	*6ccdd*
9	16	$\frac{}{1}$	$\frac{\nu}{1}$
—	ult.	fit	unde fit
11	9	${}^{\nu}\mathfrak{A}a^{\nu-2}a^{a}A$	${}^{\nu}\mathfrak{A}a^{\nu-1}a^{1}A$
—	6 a fine	his	hic
13	9	${}^{\nu}\mathfrak{A}a\ a^{1}A$	${}^{\nu}\mathfrak{A}a^{\nu-1}a^{1}A$
—	12	$p\varkappa(m+1)$	$p^{\nu}\varkappa(m+1)$
—	13 a fine	Haec	Hae
14	13	illa	illae
—	6 a fine	*Variationum* n	*Variationum summas* n
16	11	adponantur	adponatur
—	12	duabus	duobus
20	8	$\begin{pmatrix} a, & b, & c, & d, & \ldots & n \\ 1, & 2, & 3, & 4, & \ldots, & n \end{pmatrix}$	$\begin{pmatrix} a, & b, & c, & d, & \ldots & n, & \ldots \\ 1, & 2, & 3, & 4, & \ldots & n, & \ldots \end{pmatrix}$
21	6 a fine	XV	XVI
23	6 a fine	$\frac{1}{1}\mathfrak{b}^{2}B$	$\frac{1}{2}\mathfrak{b}^{2}B$
24	ult.	$\left(\frac{{}^{1}\Sigma}{1}x+\frac{{}^{2}\Sigma}{2}x+\cdots+\frac{\Sigma}{n}x^{n}\ldots\right)$	$-\left(\frac{{}^{1}\Sigma}{1}x+\frac{{}^{2}\Sigma}{2}x^{2}+\cdots+\frac{{}^{n}\Sigma}{n}x^{n}\ldots\right)$
26	9 a fine	$\begin{pmatrix} A, & B, & C, & D, & \ldots & N \\ 1, & 2, & 3, & 4, & \ldots & n \end{pmatrix}$	$\begin{pmatrix} A, & B, & C, & D, & \ldots & N, & \ldots \\ 1, & 2, & 3, & 4, & \ldots & n, & \ldots \end{pmatrix}$
27	11	nn	$n\,n$
28	14	$+\ +\ +$	$\pm\ \mp\ \pm$ (uti pag. 29. lin. 8.)
31	*Ex.* no. 3	$r = r$	$r = 3$
36	2 a fine	$\frac{2^{2}}{5^{2}\pi^{2}}$	$\frac{2^{2}}{5^{2}\pi^{2}}; d$